

Existence of Strong and Nontrivial Solutions to Strongly Coupled Elliptic Systems

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Abstract

We establish the existence of strong solutions to a class of nonlinear strongly coupled and uniform elliptic systems consisting of more than two equations. The existence of nontrivial and non constant solutions (or pattern formations) will also be studied.

1 Introduction

In this paper, we study the existence of *strong* solutions and *other nontrivial* solutions to the following nonlinear *strongly coupled* and *nonregular* but *uniform* elliptic system

$$\begin{cases} -\operatorname{div}(A(u, Du)) = \hat{f}(u, Du) \text{ in } \Omega, \\ u \text{ satisfies Dirichlet or Neumann boundary conditions on } \partial\Omega. \end{cases} \quad (1.1)$$

Here, Ω is a bounded domain with smooth boundary $\partial\Omega$ in \mathbb{R}^n , $n \geq 2$. A typical point in \mathbb{R}^n is denoted by x . The k -order derivatives of a vector valued function

$$u(x) = (u_1(x), \dots, u_m(x)) \quad m \geq 2$$

are denoted by $D^k u$. $A(u, Du)$ is a *full* matrix $m \times n$ and $\hat{f} : \mathbb{R}^m \times \mathbb{R}^{nm} \rightarrow \mathbb{R}^m$. Also, for a vector or matrix valued function $f(u, \zeta)$, $u \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^d$, its partial derivatives will be denoted by f_u, f_ζ .

Throughout this paper, we always assume the following on the diffusion matrix $A(u, Du)$.

- A)** $A(u, \zeta)$ is C^1 in $u \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^{nm}$. There are a constant $C_* > 0$ and a nonnegative scalar C^1 function $\lambda(u)$ such that for any $u \in \mathbb{R}^m$ and $\zeta, \xi \in \mathbb{R}^{nm}$

$$\lambda(u)|\zeta|^2 \leq \langle A_\zeta(u, \zeta)\xi, \xi \rangle \text{ and } |A_\zeta(u, \zeta)| \leq C_*\lambda(u). \quad (1.2)$$

Moreover, there are positive constants C, λ_0 such that $\lambda(u) \geq \lambda_0$ and $|A_u(u, \zeta)| \leq C|\lambda_u(u)||\zeta|$. In addition, $A(u, 0) = 0$ for all $u \in \mathbb{R}^m$.

The first condition in (1.2) is to say that the system (1.1) is elliptic. If $\lambda(u)$ is also bounded from above by a constant for all $u \in \mathbb{R}^m$, we say that A is regular elliptic. Otherwise, A is uniform elliptic if (1.2) holds.

Furthermore, the constant C_* in (1.2) concerns the ratio between the largest and smallest eigenvalues of A_ζ . We assume that these constants are not too far apart in the following sense.

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SG) (The spectral gap condition) $C_* < (n - 2)/(n - 4)$ if $n > 4$.

We note that if SG) is somewhat violated, i.e. C_* is large, then examples of blowing up in finite time can occur for the corresponding parabolic systems (see [2]). Of course, this condition is void in many applications when we usually have $n \leq 4$.

By a *strong solution* of (1.1) we mean a vector valued function $u \in W^{2,p}(\Omega, \mathbb{R}^m)$ for any $p > 1$ that solves (3.1) a.e. in Ω and $Du \in C^\alpha(\Omega, \mathbb{R}^m)$, $\alpha \in (0, 1)$.

As usual, $W^{k,p}(\Omega, \mathbb{R}^m)$, where k is an integer and $p \geq 1$, denotes the standard Sobolev spaces whose elements are vector valued functions $u : \Omega \rightarrow \mathbb{R}^m$ with finite norm

$$\|u\|_{W^{k,p}(\Omega, \mathbb{R}^m)} = \|u\|_{L^p(\Omega, \mathbb{R}^m)} + \sum_{i=1}^k \|D^i u\|_{L^p(\Omega, \mathbb{R}^{im})}.$$

Similarly, $C^{k,\alpha}(\Omega, \mathbb{R}^m)$ denotes the space of (vector valued) functions u on Ω such that $D^l u$, $l = 0, \dots, k$, are Hölder continuous with exponent $\alpha \in (0, 1)$. If the range \mathbb{R}^m is understood from the context we will usually omit it from the above notations.

The system (1.1) occurs in many applications concerning steady states of diffusion processes with cross diffusion taken into account, i.e. $A(u, Du)$ is a full matrix (see [11] and the reference therein). In the last few decades, there are many studies of (1.1) under the main assumption that its solutions are bounded. The lack of maximum principles for systems of more than one equations has limited the range of application of those results. Occasionally, works in this direction usually tried to establish L^∞ bounds for solutions via ad hoc techniques and thus imposed restrictive assumptions on the structural conditions of the systems. On the other hand, even if L^∞ boundedness of solutions were known, counterexamples in [10] showed that this does not suffice to guarantee higher regularity of the solutions.

Our first goal is to establish the existence of a strong solution to the general (1.1) when its L^∞ boundedness is not available. Since the system is not variational and comparison principles are generally unavailable, techniques in variational methods and monotone dynamical systems are not applicable here. Fixed point index theories will then be more appropriate to study the existence of solutions to (1.1). However, it is well known that the main ingredients of this approach are: 1) to define compact map T , whose fixed points are solutions to (1.1), on some appropriate Banach space \mathbf{X} ; 2) to show that the Leray Schauder fixed point index of T is nonzero. The second part requires some uniform estimates of the fixed points of T and regularity properties of solutions to (1.1). Those are the fundamental and most technical problems in the theory of partial differential equations. In this work, we will show that the crucial regularity property can be obtained if we know a priori that the solutions are VMO or BMO in the case of large self diffusion. We will show that the result applies to the generalized SKT systems ([25]) when the dimension of Ω is less than 5.

To this end, and throughout this work, we will impose the following structural conditions on the reaction term \hat{f} .

F) There exists a constant C such that for any C^1 functions $u : \Omega \rightarrow \mathbb{R}^m$ and $p : \Omega \rightarrow \mathbb{R}^{mn}$ satisfying $|Du| \leq |p|$ the following holds.

$$|D\hat{f}(u, p)| \leq C[\lambda(u)|Dp| + |\lambda_u(u)||Du||p| + \lambda(u)||p||]. \quad (1.3)$$

In addition, we can see that F) typically holds if there exist a constant C and a function $f(u)$ which is C^1 in u such that

$$|\hat{f}(u, p)| \leq C\lambda(u)|p| + f(u), \quad |f_u(u)| \leq C\lambda(u). \quad (1.4)$$

For simplicity, we are assuming a linear growth in p on \hat{f} . In fact, the main existence results in this work allow the following nonlinear growth for \hat{f}

$$|\hat{f}(u, p)| \leq C\lambda(u)|p|^\alpha + f(u) \quad \text{for some } \alpha \in [1, 2),$$

and such that

$$|D\hat{f}(u, p)| \leq C[\lambda(u)|p|^{\alpha-1}|Dp| + |\lambda_u(u)||Du||p|^\alpha + \lambda(u)|p|^\alpha].$$

The proof is similar with minor modifications (see Remark 2.7).

To establish the existence of a strong solution, we embed (1.1) in a suitable family of systems with $\sigma \in [0, 1]$

$$\begin{cases} -\operatorname{div}(\hat{A}_\sigma(U, DU)) = \hat{F}_\sigma(U, DU) \text{ in } \Omega, \\ \text{Homogeneous Dirichlet or Neumann boundary conditions on } \partial\Omega. \end{cases} \quad (1.5)$$

The data $\hat{A}_\sigma, \hat{F}_\sigma$ satisfy A), F) and SG) with the same set of constants. We then consider a family of compact maps $T(\sigma, \cdot)$ associated to the above systems and use a homotopy argument to compute the fixed point index of T . Again, the key point is to establish some uniform estimates of the fixed points of $T(\sigma, \cdot)$ and regularity properties of their fixed points.

The uniform estimates for Hölder norms and then higher norms of solutions to the above systems come from the crucial and technical Proposition 2.1 in Section 2 which shows that one needs only a uniform control of the $W^{1,2}(\Omega)$ and $VMO(\Omega)$ norms of (unbounded) strong solutions to the systems. Roughly speaking, we assume that for any given $\mu_0 > 0$ there is a positive R_{μ_0} for which the strong solutions to the systems in (1.5) satisfy

$$\Lambda^2 \sup_{x_0 \in \Omega} \|U\|_{BMO(B_{x_0})}^2 \leq \mu_0. \quad (1.6)$$

The proof of this result relies on a combination of a local weighted Gagliardo-Nirenberg inequality which is proved in our recent work [17] (see also [14]) and a new iteration argument using decay estimates. This technique was used in our work [17] to establish the global existence of solutions to strongly coupled parabolic systems. The proof for the elliptic case in this paper is somewhat simpler and requires less assumptions but needs some subtle modifications. For the sake of completeness and the convenience of the readers we present the details.

The fact that *bounded* weak VMO solutions to *regular* elliptic systems are Hölder continuous is now well known (see [8]). Here, using a completely different approach, we deal with *unbounded strong* VMO solutions and our Proposition 2.1 applies to *nonregular* systems. Eventually, we obtain that the strong solutions to the systems of (1.5) are uniformly Hölder continuous. Desired uniform estimates for higher norms of the solutions then follow.

Once this technical result is established, our first main result in Section 3, Theorem 3.2, then shows that (1.1) has a strong solution if the strong solutions of (1.5) are uniformly bounded in $W^{1,2}(\Omega)$ and $VMO(\Omega)$.

We present some examples in applications where Theorem 3.2 can apply. The main theme in these examples is to establish the uniform boundedness of the solutions to (1.1) in $W^{1,n}(\Omega)$, so that the solutions are in $W^{1,2}(\Omega)$ and $VMO(\Omega)$. In fact, under suitable assumptions, which occur in many mathematical models in biology and ecology, on the structural of (1.1) we will show that it is sufficient to control the very weak L^1 norms of the solutions if the dimension $n \leq 4$. Typical example in applications are the generalized SKT models (see [25]) consisting of more than 2 equations and allowing arbitrary growth conditions in the diffusion and reaction terms (see Corollary 3.10). For $n = 2$ our Corollary 3.9 generalizes a result of [21] where $A(u, Du)$ was assumed to be independent of u .

Next, we will discuss the existence of nontrivial solutions in Section 4. We now see that Theorem 3.2 establishes the existence of a strong solution in \mathbf{X} to (1.1). However, this result provides no interesting information if some 'trivial' or 'semi trivial' solutions, which are solutions to a subsystem of (1.1), are obviously guaranteed by other means. We will be interested in finding other nontrivial solutions to (1.1) and the uniform estimates in Section 3 still play a crucial role here. Although many results in this section, in particular the abstract results in Section 4.1, can apply to the general (1.1) we restrict ourselves to the system

$$\begin{cases} -\operatorname{div}(A(u)Du) = \hat{f}(u) & \text{in } \Omega, \\ \text{Homogenous Dirichlet or Neumann boundary conditions} & \text{on } \partial\Omega. \end{cases} \quad (1.7)$$

This problem is the prototype of a general class of nonlinear elliptic systems which arise in numerous applications, where u usually denotes population/chemical density vector of species/agents. Therefore, we will also be interested in finding *positive* solutions of this system, i.e. those are in the positive cone

$$\mathbf{P} := \{u \in \mathbf{X} : u = (u_1, \dots, u_m), u_i(x) \geq 0 \forall x \in \Omega\}.$$

Under suitable assumptions on \hat{f} , we will show that T can be defined as a map on a bounded set of \mathbf{P} into \mathbf{P} , i.e. T is a positive map.

If $\hat{f}(0) = 0$ then (1.7) has the *trivial solution* $u = 0$. A solution u is a *semi trivial* solution if some components of u are zero. Roughly speaking, we decompose $\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2$, accordingly $\mathbf{P} = \mathbf{P}_1 \oplus \mathbf{P}_2$ with \mathbf{P}_i being the positive cone of \mathbf{X}_i , and write an element of \mathbf{X} as (u, v) with $u \in \mathbf{X}_1$, $v \in \mathbf{X}_2$. Then $w = (u, 0)$, with $u > 0$, is a semi trivial positive fixed point if w is a fixed point of T in \mathbf{P} and u is a fixed point of $T|_{\mathbf{P}_1}$, the restriction of T to \mathbf{P}_1 . We then show that the local indices of T at these semi fixed points are solely determined by those of $T|_{\mathbf{P}_1}$ at \mathbf{P}_2 -stable fixed points.

The existence of nontrivial solutions then follows if the sum of the local fixed point indices at trivial and semi trivial solutions does not add up to the fixed point index of T in \mathbf{P} . Several results on the structure of (1.1) will be given to show that this will be the case.

Finally, in Section 5 if Neumann boundary conditions are considered then it could happen that a nontrivial and *constant* solution of (1.7) exists and solves $\hat{f}(u) = 0$. In this case, the conclusion in the previous section does not provide useful information. We are then interested in finding nontrivial *nonconstant* solutions to (1.7). The results in this section greatly improve those in [12, 16], which dealt only with systems of two equations, and establish the effect of cross diffusions in 'pattern formation' problems in mathematical

biology and chemistry. Besides the fact that our results here can be used for large systems, the analysis provides a systematic way to study pattern formation problems. Further studies and examples will be reported in our forthcoming paper [18]. We conclude the paper by presenting a simple proof of the fact that nonconstant solutions do not exist if the diffusion is sufficiently large.

2 A-priori estimates in $W^{1,p}(\Omega)$ and Hölder continuity

In this section we will establish key estimates for the proof of our main theorem Theorem 3.2 asserting the existence of strong solutions. Throughout this section, we consider two vector valued functions U, W from Ω into \mathbb{R}^m and solve the following system

$$-\operatorname{div}(A(W, DU)) = \hat{f}(W, DU). \quad (2.1)$$

We will consider the following assumptions on U, W in (2.1).

U.0) A, \hat{f} satisfy A), F) and SG) with $u = W$ and $\zeta = DU$.

U.1) $U \in W^{2,2}(\Omega) \cap C^1(\Omega)$ and $W \in C^1(\Omega)$. On the boundary $\partial\Omega$, U satisfies Neumann or Dirichlet boundary conditions.

U.2) There is a constant C such that $|DW| \leq C|DU|$ a.e. in Ω .

U.3) The following number is finite:

$$\Lambda = \sup_{W \in \mathbb{R}^m} \frac{|\lambda_W(W)|}{\lambda(W)}. \quad (2.2)$$

In the sequel, we will fix a number $q_0 > 1$ if $n \leq 4$ and, otherwise, $q_0 > (n-2)/2$ such that

$$\frac{2q_0 - 2}{2q_0} = \delta_{q_0} C_*^{-1} \text{ for some } \delta_{q_0} \in (0, 1). \quad (2.3)$$

Such numbers q_0, δ_{q_0} always exist if A) and SG) hold. In fact, if $n \leq 4$ we choose $q_0 > 1$ and sufficiently close to 1; if $n > 4$, by our assumption SG), we have $\frac{n-4}{n-2} < C_*^{-1}$ and we can choose $q_0 > (n-2)/2$ and q_0 is sufficiently close to $(n-2)/2$.

The main result of this section shows that if $\|U\|_{BMO(B_R(x_0) \cap \Omega)}$ is sufficiently small when R is uniformly small then for some $p > n$ $\|DU\|_{L^p(\Omega)}$ can be controlled.

Proposition 2.1 *Suppose that U.0)-U.3) hold. Assume that there exists $\mu_0 \in (0, 1)$, which is sufficiently small, in terms of the constants in A) and F), such that the following holds.*

D) *there is a positive R_{μ_0} such that*

$$\Lambda^2 \sup_{x_0 \in \bar{\Omega}} \|U\|_{BMO(B_{\mu_0}(x_0) \cap \Omega)}^2 \leq \mu_0, \quad (2.4)$$

Then there are $q > n/2$ and a constant C depending on the constants in U.0)-U.3), q, R_{μ_0} , the geometry of Ω and $\|DU\|_{L^2(\Omega)}$ such that

$$\int_{\Omega} |DU|^{2q} dx \leq C. \quad (2.5)$$

In particular, if U is also in $L^1(\Omega)$ then U belongs to $C^\alpha(\Omega)$ for some $\alpha > 0$ and its norm is bounded by a similar constant C as in (2.5).

The dependence of C in (2.5) on the geometry of Ω is in the following sense: Let μ_0 be as in D). We can find balls $B_{R_{\mu_0}}(x_i)$, $x_i \in \bar{\Omega}$, such that

$$\bar{\Omega} \subset \cup_{i=1}^{N_{\mu_0}} B_{R_{\mu_0}}(x_i), \quad (2.6)$$

then C in (2.5) also depends on the number N_{μ_0} .

The proof of Proposition 2.1 relies on local estimates for the integral of $|DU|$ in finitely many balls $B_R(x_i)$ with sufficiently small radius R to be determined by the condition D). We will establish local estimates for DU in these balls and then add up the results to obtain the global estimate (2.5).

2.1 Local Gagliardo-Nirenberg inequalities involving BMO norms

We first present Lemma 2.4, one of our main ingredients in the the proof of Proposition 2.1. This lemma is a simple consequence of the following local weighted Gagliardo-Nirenberg inequality which is proved in our recent work [17]. In order to state the assumption for that inequality, we recall some well known notions from Harmonic Analysis. For $\gamma \in (1, \infty)$ we say that a nonnegative locally integrable function w belongs to the class A_γ or w is an A_γ weight if the quantity

$$[w]_\gamma := \sup_{B_R(y) \subset \Omega} \left(\int_{B_R(y)} w dx \right) \left(\int_{B_R(y)} w^{1-\gamma'} dx \right)^{\gamma-1} \text{ is finite.} \quad (2.7)$$

Here, $\gamma' = \gamma/(\gamma - 1)$. For more details on these classes we refer the readers to [6, 22, 26].

We proved in [17] the following result.

Lemma 2.2 [17, Lemma 2.2] *Let $u, U : \Omega \rightarrow \mathbb{R}^m$ be vector valued functions with $u \in C^1(\Omega)$, $U \in C^2(\Omega)$ and $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a C^1 function such that*

GN) $\Phi(u)^{\frac{2}{p+2}}$ belongs to the $A_{\frac{p}{p+2}+1}$ class.

For any ball B_t in Ω we set

$$I_1(t) := \int_{B_t} \Phi^2(u) |DU|^{2p+2} dx, \quad \hat{I}_1(t) := \int_{B_t} \Phi^2(u) |Du|^{2p+2} dx, \quad (2.8)$$

$$\bar{I}_1(t) := \int_{B_t} |\Phi_u(u)|^2 (|DU|^{2p+2} + |Du|^{2p+2}) dx, \quad (2.9)$$

and

$$I_2(t) := \int_{B_t} \Phi^2(u) |DU|^{2p-2} |D^2U|^2 dx. \quad (2.10)$$

Consider any ball B_s concentric with B_t , $0 < s < t$, and any nonnegative C^1 function ψ such that $\psi = 1$ in B_s and $\psi = 0$ outside B_t . Then, for any $\varepsilon > 0$ there are positive constants $C_\varepsilon, C_{\varepsilon, \Phi}$, depending on ε and $[\Phi^{\frac{2}{p+2}}(u)]_{\frac{p}{p+2}+1}$, such that

$$\begin{aligned} I_1(s) &\leq \varepsilon [I_1(t) + \hat{I}_1(t)] + C_{\varepsilon, \Phi} \|U\|_{BMO(B_t)}^2 [\bar{I}_1(t) + I_2(t)] \\ &\quad + C_\varepsilon \|U\|_{BMO(B_t)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} |\Phi|^2(u) |DU|^{2p} dx. \end{aligned} \quad (2.11)$$

Remark 2.3 By approximation, see [24], the lemma also holds for $u \in W^{1,2}(\Omega)$ and $U \in W^{2,2}(\Omega)$ provided that the quantities I_1, I_2 and \hat{I}_1 defined in (2.8)-(2.10) are finite.

If $\Phi \equiv 1$ then $\bar{I}_1 \equiv 0$ and we can take $u = U$. The condition GN) is clearly satisfied as $[\Phi]_\gamma = 1$ for all $\gamma > 1$ (see (2.7)) and $\Phi_u = 0$. It is then clear that we have the following special version of the above lemma.

Lemma 2.4 Let $U : \Omega \rightarrow \mathbb{R}^m$ be a vector valued function in $C^2(\Omega)$. For any ball B_t in Ω we set

$$I_1(t) := \int_{B_t} |DU|^{2p+2} dx, \quad I_2(t) := \int_{B_t} |DU|^{2p-2} |D^2U|^2 dx. \quad (2.12)$$

Consider any ball B_s concentric with B_t , $0 < s < t$, and any nonnegative C^1 function ψ such that $\psi = 1$ in B_s and $\psi = 0$ outside B_t . Then, for any $\varepsilon > 0$ there is a positive constant C_ε such that

$$\begin{aligned} I_1(s) &\leq \varepsilon I_1(t) + C_\varepsilon \|U\|_{BMO(B_t)}^2 I_2(t) \\ &\quad + C_\varepsilon \|U\|_{BMO(B_t)} \sup_{x \in B_t} |D\psi(x)|^2 \int_{B_t} |DU|^{2p} dx. \end{aligned} \quad (2.13)$$

2.2 The proof of Proposition 2.1

In the proof, we will only consider the case when $B_R(x_i) \subset \Omega$. The boundary case ($x_i \in \partial\Omega$) is similar, using the fact that $\partial\Omega$ is smooth and a reflection argument to extend the function U or DU outside Ω , see Remark 2.9 and Remark 2.10.

For any $x_0 \in \bar{\Omega}$ and $t > 0$ we will denote $B_t(x_0) = B_t(x_0) \cap \Omega$. For $q \geq 1$ we introduce the following quantities.

$$\mathcal{B}_q(t, x_0) = \int_{B_t(x_0)} |DU|^{2q+2} dx, \quad (2.14)$$

$$\mathcal{H}_q(t, x_0) = \int_{B_t(x_0)} |DU|^{2q-2} |D^2U|^2 dx, \quad (2.15)$$

$$\mathcal{G}_q(t, x_0) = \int_{B_t(x_0)} |DU|^{2q} dx. \quad (2.16)$$

In the rest of this section, let us fix a point x_0 in Ω and drop x_0 in the notations (2.14)-(2.16).

For any s, t such that $0 < s < t \leq R$ let ψ be a cutoff function for two balls B_s, B_t centered at x_0 . That is, ψ is nonnegative, $\psi \equiv 1$ in B_s and $\psi \equiv 0$ outside B_t with $|D\psi| \leq 1/(t-s)$.

We first have the following local energy estimate.

Lemma 2.5 *Asume U.0)-U.3). Assume that $q \geq 1$ satisfies the condition*

$$\frac{2q-2}{2q} = \delta_q C_*^{-1} \text{ for some } \delta_q \in (0, 1). \quad (2.17)$$

There is a constant $C_1(q)$ depending also on the constants in A) and F) such that

$$\mathcal{H}_q(s) \leq C_1(q) \left[\Lambda^2 \mathcal{B}_q(t) + \frac{1}{(t-s)^2} \mathcal{G}_q(t) \right] \quad 0 < s < t \leq R. \quad (2.18)$$

Proof: By the assumption U.1), we can formally differentiate (2.1) with respect to x , more precisely we can use difference quotients (see Remark 2.6), to get the weak form of

$$-\operatorname{div}(A_\zeta(W, DU)D^2U + A_W(W, DU)DW DU) = D\hat{f}(W, DU). \quad (2.19)$$

We denote $\beta(W) = \lambda^{-1}(W)$. Testing (2.19) with $\phi = \beta(W)|DU|^{2q-2}DU\psi^2$, which is legitimate since \mathcal{H}_q is finite, integrating by parts in x and rearranging, we have

$$\int_{\Omega} \langle A_\zeta(W, DU)D^2U + A_W(W, DU)DW DU, D\phi \rangle dx = \int_Q \langle D\hat{f}(W, DU), \phi \rangle dx. \quad (2.20)$$

For simplicity, we will assume in the proof that $\hat{f} \equiv 0$. As $D\phi = I_0 + I_1 + I_2$ with

$$I_0 := \beta(W)D(|DU|^{2q-2}DU)\psi^2, \quad I_1 := |DU|^{2q-2}DU\beta_W DW \psi^2,$$

and $I_2 := 2\beta(W)|DU|^{2q-2}DU\psi D\psi$, we can rewrite (2.20) as

$$\begin{aligned} & \int_{\Omega} \beta(W) \langle A_\zeta(W, DU)D^2U, D(|DU|^{2q-2}DU)\psi^2 \rangle dx \\ &= - \int_{\Omega} [\langle A_\zeta(W, DU)D^2U, I_1 + I_2 \rangle + \langle A_W(W, DU)DW, D\phi \rangle] dx. \end{aligned} \quad (2.21)$$

Let us first consider the integral on the left hand side. By U.0) and the uniform ellipticity of $A_\zeta(W, DU)$, we can find a constant C_* such that $|A_\zeta(W, DU)\zeta| \leq C_*\lambda(W)|\zeta|$. On the other hand, By (2.17), $\alpha = 2q - 2$ satisfies

$$\frac{\alpha}{2 + \alpha} = \frac{2q-2}{2q} = \delta_q C_*^{-1} = \delta_q \frac{\lambda(W)}{C_*\lambda(W)}.$$

By [2, Lemma 2.1], or [14, Lemma 6.2], for such α, q there is a positive constant $C(q)$ such that

$$\langle A_\zeta(W, DU)D^2U, D(|DU|^{2q-2}DU) \rangle \geq C(q)\lambda(W)|DU|^{2q-2}|D^2U|^2. \quad (2.22)$$

Because $\beta(W)\lambda(W) = 1$, we then obtain from (2.21)

$$\begin{aligned} & C_0(q) \int_Q |DU|^{2q-2}|D^2U|^2\psi^2 dx \\ & \leq - \int_{\Omega} [\langle A_\zeta(W, DU)D^2U, I_1 + I_2 \rangle + \langle A_W(W, DU)DW, D\phi \rangle] dx. \end{aligned} \quad (2.23)$$

The terms I_1, I_2 in the integrands on the right hand side of (2.23) can be easily handled by using the fact that $|D\psi| \leq 1/(t-s)$ and the assumption A) which gives

$$|A_\zeta(W, DU)| \leq C|\lambda(W)| \text{ and } |A_W(W, DU)| \leq C|\lambda_W(W)||DU|.$$

We also note that $|\beta_W(W)| = \lambda^{-2}(W)|\lambda_W(W)| \leq \lambda^{-1}(W)\mathbf{\Lambda}$ (see (2.2)).

Concerning the first integrand on the right of (2.21), using the definition of I_i and Young's inequality, for any $\varepsilon > 0$ we can find a constant $C(\varepsilon)$ such that

$$|\langle A_\zeta(W, DU)D^2U, I_1 \rangle| \leq \varepsilon|DU|^{2q-2}|D^2U|^2\psi^2 + C(\varepsilon)\mathbf{\Lambda}^2|DW|^2|DU|^{2q}\psi^2,$$

$$|\langle A_\zeta(W, DU)D^2U, I_2 \rangle| \leq \varepsilon|DU|^{2q-2}|D^2U|^2\psi^2 + C(\varepsilon)|DU|^{2q}|D\psi|^2.$$

Similarly, for the second integrand on the right of (2.21) we have

$$|\langle A_W(W, DU)DW, I_0 \rangle| \leq \varepsilon|DU|^{2q-2}|D^2U|^2\psi^2 + C(\varepsilon)\mathbf{\Lambda}^2|DW|^2|DU|^{2q}\psi^2,$$

$$|\langle A_W(W, DU)DW, I_1 \rangle| \leq C\mathbf{\Lambda}^2|DW|^2|DU|^{2q}\psi^2,$$

$$|\langle A_W(W, DU)DW, I_2 \rangle| \leq C\mathbf{\Lambda}^2|DW|^2|DU|^{2q}\psi^2 + C|DU|^{2q}|D\psi|^2.$$

Choosing ε sufficiently small, we then obtain from the above inequalities and the assumption $|DW| \leq C|DU|$ that

$$\int_{B_s} |DU|^{2q-2}|D^2U|^2 dx \leq C_1\mathbf{\Lambda}^2 \int_{B_t} |DU|^{2q+2} dx + C_1 \frac{1}{(t-s)^2} \int_{B_t} |DU|^{2q} dx.$$

From the notations (2.14)-(2.16), the above estimate gives the lemma. ■

Remark 2.6 For $i = 1, \dots, n$ and $h \neq 0$ we denote by $\delta_{i,h}$ the difference quotient operator $\delta_{i,h}u = h^{-1}(u(x + he_i) - u(x))$, with e_i being the unit vector of the i -th axis in \mathbb{R}^n . We then apply $\delta_{i,h}$ to the system for U and then test the result with $|\delta_{i,h}U|^{2q-2}\delta_{i,h}U\psi^2$. The proof then continues to give the desired energy estimate by letting h tend to 0.

Remark 2.7 If $\hat{f} \neq 0$ then there is an extra term $|D\hat{f}(W, DU)||DU|^{2q-1}\psi^2$ in (2.23). This term will give rise to similar terms in the proof. Indeed, by (1.3) in F) with $u = W$ and $p = DU$,

$$|D\hat{f}(W, DU)| \leq C[\lambda(W)|D^2U| + |\lambda_W(W)||DW||DU| + \lambda(W)|DU|].$$

As $\beta(W) = \lambda^{-1}(W)$ and $|DW| \leq C|DU|$, for any $\varepsilon > 0$ we can use Young's inequality and the definition of $\mathbf{\Lambda}$ to find a constant $C(\varepsilon)$ such that

$$\begin{aligned} |D\hat{f}(W, DU)|\beta(W)|DU|^{2q-1} &\leq C[|D^2U||DU|^{2q-1} + \mathbf{\Lambda}|DU|^{2q+1} + |DU|^{2q}] \\ &\leq \varepsilon|DU|^{2q-2}|D^2U|^2 + \mathbf{\Lambda}^2|DU|^{2q+2} + C(\varepsilon)|DU|^{2q}. \end{aligned}$$

Choosing $\varepsilon > 0$ sufficiently small, we then see that the proof can continue to obtain the energy estimate (2.18).

Similarly, if we can allow \hat{f} to have nonlinear growth in DU by replacing (1.3) with

$$|D\hat{f}(u, p)| \leq C[\lambda(u)|p|^{\alpha-1}|Dp| + |\lambda_u(u)||Du||p|^\alpha + \lambda(u)|p|^\alpha] \quad \text{for some } \alpha \in [1, 2).$$

Then $|D\hat{f}(W, DU)|\beta(W)|DU|^{2q-1}$ can be estimated by

$$C[|D^2U||DU|^{2q+\alpha-2} + \mathbf{\Lambda}|DU|^{2q+\alpha} + |DU|^{2q+\alpha-1}]$$

Again, by Young's inequality and $q \geq 1$ and $\alpha < 2$, it is not difficult to see that there is some exponent $\gamma > 0$ depending on α such that the above is bounded by

$$\varepsilon|DU|^{2q-2}|D^2U|^2 + (\mathbf{\Lambda}^2 + \varepsilon)|DU|^{2q+2} + C(\varepsilon)|DU|^{2q} + C(\varepsilon)(\mathbf{\Lambda}^\gamma + 1),$$

and the proof can continues.

Remark 2.8 Inspecting our proof here and the proof of [14, Lemma 6.2], we can see that the constant $C(q)$ in (2.22) is decreasing in q and hence $C_1(q)$ is increasing in q . Note also that this is the only place where we need (2.17).

Remark 2.9 We discuss the case when the centers of B_ρ, B_R are on the boundary $\partial\Omega$. We assume that U satisfies the Neumann boundary condition on $\partial\Omega$. By flattening the boundary we can assume that $B_R \cap \Omega$ is the set

$$B^+ = \{x : x = (x_1, \dots, x_n) \text{ with } x_n \geq 0 \text{ and } |x| < R\}.$$

For any point $x = (x_1, \dots, x_n)$ we denote by \bar{x} its reflection across the plane $x_n = 0$, i.e., $\bar{x} = (x_1, \dots, -x_n)$. Accordingly, we denote by B^- the reflection of B^+ . For a function u given on B_+ we denote its even reflection by $\bar{u}(x) = u(\bar{x})$ for $x \in B^-$. We then consider the even extension of \hat{u} in $B = B^+ \cup B^-$

$$\hat{u}(x) = \begin{cases} u(x) & \text{if } x \in B^+, \\ \bar{u}(x) & \text{if } x \in B^-. \end{cases}$$

With these notations, for $x \in B^+$ we observe that $\operatorname{div}_x(D_x U) = \operatorname{div}_{\bar{x}}(D_{\bar{x}} \bar{U})$ and $D_x W D_x U = D_{\bar{x}} \bar{W} D_{\bar{x}} \bar{U}$. Therefore, it is easy to see that \hat{U} satisfies in B a system similar to the one for U in B^+ . Thus, the proof can apply to \hat{U} to obtain the same energy estimate near the boundary.

Remark 2.10 For the Dirichlet boundary condition we make use of the odd reflection $\bar{u}(x) = -u(\bar{x})$ and then define \hat{u} as in Remark 2.9. Since $D_{x_i} U = 0$ on $\partial\Omega$ if $i \neq n$, we can test the system (2.19), obtained by differentiating the system of U with respect to x_i , with $|D_{x_i} U|^{2q-2} D_{x_i} U \psi^2$ and the proof goes as before because no boundary integral terms appear in the calculation. We need only consider the case $i = n$. We observe that $D_{x_n} \hat{U}$ is the even extension of $D_{x_n} U$ in B therefore \hat{U} satisfies a system similar to (2.19). The proof then continues.

Next, let us recall the following elementary iteration result in [17] (which is a consequence of [8, Lemma 6.1, p.192]).

Lemma 2.11 *Let F, G, g, h be bounded nonnegative functions in the interval $[\rho, R]$ with g, h being increasing. Assume that for $\rho \leq s < t \leq R$ we have*

$$F(s) \leq \varepsilon_0[F(t) + G(t)] + [(t-s)^{-\alpha}g(t) + h(t)], \quad (2.24)$$

$$G(s) \leq C[F(t) + (t-s)^{-\alpha}g(t) + h(t)] \quad (2.25)$$

with $C \geq 0, \alpha, \varepsilon_0 > 0$.

If $2C\varepsilon_0 < 1$ then there is constant $c(C, \alpha, \varepsilon_0)$ such that

$$F(s) + G(s) \leq c(C, \alpha, \varepsilon_0)[(t-s)^{-\alpha}g(t) + h(t)] \quad \rho \leq s < t \leq R. \quad (2.26)$$

We are now ready to give the proof of the main result, Proposition 2.1, of this section.

Proof: For any given $R_0, \varepsilon > 0$, multiplying (2.13) by Λ^2 and using the notations (2.14)-(2.16), we can find a constant C_ε such that

$$\Lambda^2 \mathcal{B}_q(s) \leq \varepsilon \Lambda^2 \mathcal{B}_q(t) + C_\varepsilon \Lambda^2 \|U\|_{BMO(B_t)}^2 \mathcal{H}_q(t) + C_\varepsilon \|U\|_{BMO(B_t)} \frac{\Lambda^2}{(t-s)^2} \mathcal{G}_q(t) \quad (2.27)$$

for all s, t such that $0 < s < t \leq R_0$.

On the other hand, let $q_0 > 1$ and satisfies (2.3). We then have

$$\frac{2q-2}{2q} < C_*^{-1} \quad \forall q \in [1, q_0].$$

Hence (2.17) of Lemma 2.5 holds for $q \in [1, q_0]$ and we obtain from (2.18) that

$$\mathcal{H}_q(s) \leq C_1(q) \Lambda^2 \mathcal{B}_q(t) + \frac{C_1(q)}{(t-s)^2} \mathcal{G}_q(t), \quad 0 < s < t \leq R_0. \quad (2.28)$$

We define

$$F(t) = \Lambda^2 \mathcal{B}_q(t), \quad G(t) = \mathcal{H}_q(t), \quad g(t) = \max\{C_1(q_0), C_\varepsilon \|U\|_{BMO(B_{R_0})} \Lambda^2\} \mathcal{G}_q(t),$$

$$\varepsilon_0 = \max\{\varepsilon, C_\varepsilon \Lambda^2 \|U\|_{BMO(B_{R_0})}^2\}.$$

It is clear that (2.27), (2.28) respectively imply (2.24) and (2.25) of Lemma 2.11 with $C = C_1(q_0)$, using the fact that (see Remark 2.8) $C_1(q)$ is increasing in q .

We first choose ε such that $2C_1(q_0)\varepsilon < 1$ and then $R_0 > 0$ such that

$$2C_1(q_0)C_\varepsilon \Lambda^2 \|U\|_{BMO(B_{R_0})}^2 < 1. \quad (2.29)$$

We thus have $2C_1(q)\varepsilon_0 < 1$ so that (2.26) of Lemma 2.11 provide a constant C_2 depending on $C_1(q_0), \varepsilon_0$ such that

$$\mathcal{H}_q(s) + \Lambda^2 \mathcal{B}_q(s) \leq \frac{C_2}{(t-s)^2} \mathcal{G}_q(t), \quad 0 < s < t \leq R_0.$$

For $t = 2s$ the above gives (if q satisfies (2.17))

$$\mathcal{H}_q(s) + \Lambda^2 \mathcal{B}_q(s) \leq \frac{C_3}{s^2} \int_{Q_{2s}} |DU|^{2q} dx \quad 0 < s \leq \frac{R_0}{2}. \quad (2.30)$$

Using this estimate for $\mathcal{B}_q(t)$ in (2.18), with $s = R_0/4$ and $t = R_0/2$ respectively, we derive

$$\mathcal{H}_q(R_1) \leq \frac{C_4}{R_1^2} \int_{B_{R_1}} |DU|^{2q} dx, \quad R_1 = \frac{R_0}{4}. \quad (2.31)$$

Now, we will argue by induction to obtain a bound for \mathcal{A}_q for some $q > n/2$. Suppose that for some $q \geq 1$ and q satisfies (2.17) we can find a constant C_q such that

$$\int_{\Omega} |DU|^{2q} dx \leq C_q, \quad (2.32)$$

and that (2.29) holds then (2.31) implies similar bound for $\mathcal{H}_q(R_1)$. We now can cover Ω by N_{R_1} balls B_{R_1} , see (2.6), and add up the estimate (2.31) for $\mathcal{H}_q(R_1)$ to obtain a constant $C(\Omega, R_1, N_{R_1}, q)$ such that

$$\int_{\Omega} |DU|^{2q-2} |D^2 U|^2 dx \leq C(\Omega, R_1, N_{R_1}, q, C_q). \quad (2.33)$$

Hence, (2.32) and (2.33) yield another constant $C(\Omega, R_1, N_{R_1}, q, C_q)$ such that

$$\int_{\Omega} |DU|^{2q} dx + \int_{\Omega} |DU|^{2q-2} |D^2 U|^2 dx \leq C(\Omega, R_1, N_{R_1}, q, C_q).$$

Therefore, the $W^{1,2}(\Omega)$ norm of $|DU|^q$ is bounded. Let $n_* = n/(n-2)$ (or any number greater than 1 if $n = 2$). By Sobolev's inequality, the above implies that there is a constant $C(\Omega, R_1, q_*)$ such that

$$\int_{\Omega} |DU|^{2qq_*} dx \leq C(\Omega, R_1, N_{R_1}, q_*) \quad \text{for any } q_* \in (1, n_*]. \quad (2.34)$$

We now see that (2.32) holds again with the exponent q being replaced by qq_* .

Of course, (2.32) holds for $q = 1$ with $C_1 = \|DU\|_{L^2(\Omega)}^2$. Hence, for suitable choice of an integer k_0 and $q_* \in (1, n_*)$ to be determined later we define $L_k = q_*^k$ and repeat the above argument, with $q = L_k$, k_0 times as long as $L_k \leq q_0$, $k = 0, 1, 2, \dots, k_0$. We then obtain from (2.34)

$$\int_{\Omega} |DU|^{2L_{k_0} n_*} dx \leq C(\|DU\|_{L^2(\Omega)}, \Omega, R_1, N_{R_1}, q_*, k_0). \quad (2.35)$$

We now determine q_* and k_0 . If $n \leq 4$ we let $k_0 = 1$ and $q_* = \min\{q_0, n_*\}$. Otherwise, if $n > 4$, it is clear that we can find $q_* \in (1, n_*)$ and an integer k_0 such that $L_{k_0} = q_*^{k_0} = q_0$. Since $q_0 > 1$ if $n \leq 4$ and $q_0 > (n-2)/2$ otherwise, it is clear that $2L_{k_0} n_* > n$ in both cases.

Therefore, (2.35) shows that (2.5) holds for $q = L_{k_0} n_* > n/2$. Since q_*, k_0 depend on q_0 , the constant in (2.35) essentially depends on the parameters in U.0)-U.3) and the geometry of Ω . The proof is complete. ■

Remark 2.12 From Remark 2.3, we can see that the conclusion of Proposition 2.1 continues to hold for $U \in W^{2,2}$ as long as the quantities (2.14)-(2.16) are finite for $q \in [1, q_0]$, q_0 is fixed in (2.3).

3 Existence of Strong Solutions

We now consider the system (1.1) in this section. Recall that

$$-\operatorname{div}(A(u, Du)) = \hat{f}(u, Du) \quad (3.1)$$

in Ω and u satisfies homogeneous Dirichlet or Neumann boundary conditions on $\partial\Omega$. Throughout this section we will assume that A, \hat{f} satisfy A), F) and SG).

To establish the existence of a strong solution, we embed the systems (3.1) in the following family of systems with $\sigma \in [0, 1]$

$$\begin{cases} -\operatorname{div}(\hat{A}_\sigma(U, DU)) = \hat{F}_\sigma(U, DU) \text{ in } \Omega, \\ U \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega. \end{cases}$$

We will introduce a family of maps $T(\sigma, \cdot)$, $\sigma \in [0, 1]$, acting in some suitable Banach space \mathbf{X} such that their fixed points are strong solutions to the above system. We then use Leray-Schauder's fixed point index theory to establish the existence of a fixed point of $T(1, \cdot)$, which is a strong solution to (3.1).

Fixing some $\alpha_0 \in (0, 1)$, we consider the Banach space

$$\mathbf{X} := C^{1, \alpha_0}(\Omega) \text{ (resp. } C^{1, \alpha_0}(\Omega) \cap C_0(\Omega)) \quad (3.2)$$

if Neumann (resp. Dirichlet) boundary conditions are assumed for (3.1).

For each $w \in \mathcal{X}$ and $\sigma \in [0, 1]$, we define

$$A_\sigma(w) = \int_0^1 \partial_2 A(\sigma w, t\sigma Dw) dt.$$

Here and in the sequel, we will also use the notations $\partial_1 g(u, \zeta)$, $\partial_2 g(u, \zeta)$ to denote the partial derivatives of a function $g(u, \zeta)$ with respect to its variables u, ζ .

Assume that there is a family of vector valued functions $\hat{f}_\sigma(U, \zeta)$ with $\sigma \in [0, 1]$, $U \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^{nm}$ such that

f.0) $\hat{f}_\sigma(U, \zeta)$ is continuous in σ and C^1 in U, ζ .

f.1) $\hat{f}_0(U, \zeta) \equiv 0$ and $\hat{f}_1(U, \zeta) = \hat{f}(U, \zeta)$ for all U, ζ .

f.2) \hat{f}_σ satisfies F) uniformly for $\sigma \in [0, 1]$. That is, there is a constant C such that for $U \in W^{2,2}(\Omega)$ and $W = \sigma U$

$$|D\hat{f}_\sigma(U, DU)| \leq C[\lambda(W)|D^2U| + |\lambda_W(W)||DW||DU| + \lambda(W)|DU|],$$

a.e. in Ω .

Let K be any constant matrix satisfying

$$\langle Ku, u \rangle \geq k|u|^2 \quad \text{for some } k > 0 \text{ and all } u \in \mathbb{R}^m. \quad (3.3)$$

For a given $w \in \mathbf{X}$ and $\sigma \in [0, 1]$ we consider the following *linear* elliptic system for u

$$\begin{cases} -\operatorname{div}(A_\sigma(w)Du) + Ku + u = \hat{f}_\sigma(w, Dw) + Kw + \sigma w & \text{in } \Omega, \\ \text{Homogeneous Dirichlet or Neumann boundary conditions} & \text{on } \partial\Omega. \end{cases} \quad (3.4)$$

From A) and (3.3) we easily see that the system

$$\begin{cases} -\operatorname{div}(A_\sigma(w)Du) + Ku + u = 0 & \text{in } \Omega \\ \text{Homogeneous Dirichlet or Neumann boundary conditions} & \text{on } \partial\Omega \end{cases}$$

has $u = 0$ as the only solution. From the theory of linear elliptic systems with Hölder continuous coefficient, (3.4) has a unique strong solution u . We then define $T(\sigma, w) = u$.

As A satisfies A), $A(\sigma U, 0) = 0$. Hence, for $\sigma \in (0, 1]$

$$A_\sigma(U)DU = \int_0^1 \partial_2 A(\sigma U, t\sigma DU) dt DU = \sigma^{-1} A(\sigma U, \sigma DU). \quad (3.5)$$

Meanwhile $A_0(U) = \partial_2 A(0, 0)$.

We now define

$$\hat{A}_\sigma(U, \zeta) = \sigma^{-1} A(\sigma U, \sigma \zeta) \quad \sigma \in (0, 1], \quad \hat{A}_0(U, \zeta) = \partial_2 A(0, 0)\zeta. \quad (3.6)$$

The fixed points of $T(\sigma, \cdot)$, defined by (3.4) with $\sigma \in [0, 1]$, are solutions the following family of systems

$$\begin{cases} -\operatorname{div}(\hat{A}_\sigma(U, DU)) = \hat{f}_\sigma(U, DU) + (\sigma - 1)U & \text{in } \Omega, \\ U \text{ satisfies homogeneous Dirichlet or Neumann BC} & \text{on } \partial\Omega. \end{cases} \quad (3.7)$$

Remark 3.1 A typical choice of \hat{f}_σ in applications is $\hat{f}_\sigma(U, \zeta) = \hat{f}(\sigma U, \sigma \zeta)$. It is not difficult to see that $\hat{f}_\sigma(U, \zeta)$ satisfies f.1)-f.2) if \hat{f} does.

3.1 Existence of Strong Solutions:

The main result of this section is the following result.

Theorem 3.2 *We assume that A, \hat{f}_σ satisfy A), f.0)-f.2) and SG) and that the following number is finite:*

$$\mathbf{\Lambda} = \sup_{W \in \mathbb{R}^m} \frac{|\lambda_W(W)|}{\lambda(W)}. \quad (3.8)$$

In addition, we assume that the following conditions hold uniformly for any solution U to (3.7).

U) *There is a constant C such that*

$$\|U\|_{W^{1,2}(\Omega)} \leq C. \quad (3.9)$$

M) for any given $\mu_0 > 0$ there is a positive R_{μ_0} for which

$$\Lambda^2 \sup_{x_0 \in \bar{\Omega}} \|U\|_{BMO(B_{x_0})}^2 \leq \mu_0. \quad (3.10)$$

Then (3.1) has at least one strong solution.

Proof: We will use Leray-Schauder's fixed point index theory to establish the existence of a fixed point of $T(1, \cdot)$, which is a strong solution to (3.1) and the theorem then follows. To this end, we will establish the facts.

- i) $T(\sigma, \cdot) : \mathbf{X} \rightarrow \mathbf{X}$ is compact for $\sigma \in (0, 1]$.
- ii) $\text{ind}(T(0, \cdot), \mathbf{B}, \mathbf{X}) = 1$ (see the definition of indices below).
- iii) A fixed point $u = T(\sigma, u)$ is a solution to (3.7). For $\sigma = 1$, such fixed points are solutions to (3.1).
- iv) There is $M > 0$, independent of $\sigma \in [0, 1]$ and K , such that any fixed point $u^{(\sigma)} \in \mathbf{X}$ of $T(\sigma, \cdot)$ satisfies $\|u^{(\sigma)}\|_{\mathbf{X}} < M$.

Once i)-iv) are established, the theorem follows from the Leray-Schauder index theory. Indeed, we let \mathbf{B} be the ball centered at 0 with radius M of \mathbf{X} and consider the Leray-Schauder indices

$$\text{ind}(T(\sigma, \cdot), \mathbf{B}, \mathbf{X}) \stackrel{\text{def}}{=} \deg(Id - T(\sigma, \cdot), \mathbf{B}, 0), \quad (3.11)$$

where the right hand side denote the Leray-Schauder degree with respect to zero of the vector field $Id - T(\sigma, \cdot)$. This number is well defined because $T(\sigma, \cdot)$ is compact (by i)) and $Id - T(\sigma, \cdot)$ does not have zero on $\partial\mathbf{B}$ (by iv)).

By the homotopy invariance of the indices, $\text{ind}(T(\sigma, \cdot), \mathbf{B}, \mathbf{X}) = \text{ind}(T(0, \cdot), \mathbf{B}, \mathbf{X})$, which is 1 because of ii). Thus, $T(\sigma, \cdot)$ has a fixed point in \mathbf{B} for all $\sigma \in [0, 1]$. Our theorem then follows from iii).

Using regularity properties of solutions to linear elliptic systems with Hölder continuous coefficients, we see that i) holds. The proof of ii) is standard (see Remark 3.3 after the proof). Next, iii) follows from the assumption on \hat{f}_1 in f.1).

Finally, the main point of the proof is iv). We have to establish a uniform estimate for the fixed points of $T(\sigma, \cdot)$ in \mathbf{X} . To check iv), let $u^{(\sigma)} \in \mathbf{X}$ be a fixed point of $T(\sigma, \cdot)$, $\sigma \in [0, 1]$. We need only consider the case $\sigma > 0$. Clearly, $u^{(\sigma)}$ solves

$$-\text{div}(A_\sigma(u^{(\sigma)})Du^{(\sigma)}) = \hat{f}_\sigma(u^{(\sigma)}, Du^{(\sigma)}) + (\sigma - 1)u^{(\sigma)}$$

so that $U = u^{(\sigma)}$ is a strong solution of (3.7). We need to show that $\|U\|_{\mathbf{X}}$ is uniformly bounded for $\sigma \in [0, 1]$.

We now denote $W = \sigma U$ and will show that Proposition 2.1 can be applied to the systems (3.7). As we assume (3.8) and $W = \sigma U$, with $u^{(\sigma)} \in \mathbf{X}$ and U is a strong solution, the conditions U.1) and U.2) are clearly verified.

We will see that U.0) is verified. Firstly, from (3.5) and the assumption that A satisfies A) and we will show that $\hat{A}_\sigma(U, \zeta)$ satisfies A) too. Indeed,

$$\langle \hat{A}_\sigma(U, \zeta), \zeta \rangle = \langle \sigma^{-1}A(\sigma U, \sigma \zeta), \zeta \rangle = \langle \sigma^{-2}A(\sigma U, \sigma \zeta), \sigma \zeta \rangle \geq \lambda(\sigma U)|\zeta|^2,$$

$$\begin{aligned}\|\hat{A}_\sigma(U, \zeta)\| &= \sigma^{-1} \|A(\sigma U, \sigma \zeta)\| \leq C_* \lambda(\sigma U) |\zeta|, \\ \left\| \frac{\partial}{\partial U} \hat{A}_\sigma(U, \zeta) \right\| &= \|\partial_1 A(\sigma U, \zeta)\| \leq \lambda_{\sigma U}(\sigma U) |\zeta|.\end{aligned}$$

Therefore \hat{A}_σ satisfies A) with $u = \sigma U$.

Secondly, from the assumption f.2) on $\hat{f}_\sigma(U, \zeta)$, satisfying F) uniformly for $\sigma \in [0, 1]$, and the fact that $\lambda(W)$ is bounded from below we see that the right hand side of (3.7) satisfies F). Thus, U.0) is satisfied for the system (3.7).

Finally, it is clear that (3.10) in the assumption M) gives the condition D) of Proposition 2.1. The assumption (3.9) of U) yields that $\|DU\|_{L^2(\Omega)}$ is bounded (see also Remark 3.4 after the proof). More importantly, the uniform bound in (3.10) then gives some positive constants $\mu_0, R(\mu_0)$ such that Proposition 2.1 applies to $U = u^{(\sigma)}, W = \sigma U$ and gives a uniform estimate for $\|u^{(\sigma)}\|_{W^{1,2q}(\Omega)}$ for some $q > n/2$ and $\sigma \in [0, 1]$. By Sobolev's imbedding theorems this shows that $u^{(\sigma)}$ is Hölder continuous with its norm uniformly bounded with respect to $\sigma \in [0, 1]$. Since A is C^1 in u , the results in [8] then imply that $Du^{(\sigma)} \in C^\alpha(\Omega)$ for any $\alpha \in (0, 1)$ and its norm is uniformly bounded by a constant independent of σ, K . We then obtain a uniform estimate for $\|u^{(\sigma)}\|_{\mathbf{X}}$ and iv) is verified.

We then see that (3.1) has a solution u in \mathbf{X} . Furthermore, [8, Chapter 10] shows that u is a strong solution. The proof is complete. ■

Remark 3.3 The map $T(0, \cdot)$ is defined by the following *linear* elliptic system with constant coefficients ($A_0 := A_0(w) = \partial_2 A(0, 0)$ and $\hat{f}_0(w, Dw) \equiv 0$)

$$-\operatorname{div}(A_0 Du) + Ku + u = Kw \quad (3.12)$$

with homogeneous Dirichlet or Neumann boundary conditions. We then consider the following family of systems, with the same boundary conditions, for $\tau \in [0, 1]$

$$-\operatorname{div}(A_0 Du) + Ku + u = \tau Kw \quad (3.13)$$

and define the maps $H(\tau, \cdot)$ on \mathbf{X} by $H(\tau, w) = u$. The fixed points u of $H(\tau, \cdot)$ satisfy (3.13) with $u = w$ so that by testing this with u and using A) and (3.3) we easily see that $u = 0$. Similarly, $H(0, \cdot) = 0$, a constant map. Thus, by homotopy, $\operatorname{ind}(H(1, \cdot), \mathbf{B}, \mathbf{X}) = \operatorname{ind}(H(0, \cdot), \mathbf{B}, \mathbf{X}) = 1$. Obviously, $T(0, \cdot) = H(1, \cdot)$ so that $\operatorname{ind}(T(0, \cdot), \mathbf{B}, \mathbf{X}) = 1$.

Remark 3.4 In applications, the assumption on the boundedness of $\|U\|_{W^{1,2}(\Omega)}$ in U) can be removed if $\lambda(u)$ has a polynomial growth in $|u|$ and $\|U\|_{L^1(\Omega)}$ is bounded uniformly. We sketch the proof here. We first observe that $\|U\|_{L^q(\Omega)}$ is uniformly bounded. In fact, by [8, Corollary 2.2] and then M), there are constants $C_q, C(q, \mu_0)$ such that for $R \leq R_{\mu_0}$

$$\left(\frac{1}{|B_R|} \int_{B_R} |U - U_R|^q dx \right)^{\frac{1}{q}} \leq C_q \|U\|_{BMO(B_R)} \leq C(q, \mu_0). \quad (3.14)$$

We easily deduct from the above estimate that there is a constant C depending on $\mu_0, R_{\mu_0}, \mathbf{A}$ and $\|U\|_{L^1(\Omega)}$ such that $\|U\|_{L^q(B_{R_{\mu_0}})} \leq C$.

For $W = \sigma U$ we now test the system (2.1) with $\psi = (U - U_{2R})\phi^2$, where ϕ is a cut off function for B_R, B_{2R} and satisfies $|D\phi| \leq CR^{-1}$. We get

$$\int_{B_{2R}} \lambda(W) |DU|^2 \phi^2 dx \leq C \int_{B_{2R}} (|A(W, DU)| |U - U_{2R}| |D\phi| \phi + |\hat{f}(W, DU)| |\psi|) dx.$$

Inspired by the condition f.2), if λ has a polynomial growth then we can assume that

$$|\hat{f}(W, DU)| \leq C\lambda(W)|DU| + C\lambda(W)|U|.$$

Thus, we can use Young's inequality to obtain the following Caccioppoli type estimate

$$\int_{B_R} \lambda(W) |DU|^2 dx \leq C \int_{B_{2R}} [R^{-2}\lambda(W) + \lambda(W)|U|] |U - U_{2R}|^2 dx. \quad (3.15)$$

Let $R = R_{\mu_0}/2$. If $\lambda(W)$ has a polynomial growth in W and $|W| \leq |U|$, we can apply Young's inequality to the right hand side to see that it is bounded in terms of $R_{\mu_0}, \|U\|_{L^q(B_{R_{\mu_0}})}^q$ and the constant $C(q, \mu_0)$ in (3.14). Using a finite covering of Ω and the fact that $\lambda(W)$ is bounded from below, we add the above inequalities to obtain a uniform bound for $\|DU\|_{L^2(\Omega)}$. Hence, the assumption U) can be removed in this case.

Remark 3.5 We applied Proposition 2.1 to strong solution in the space \mathbf{X} so that U, DU are bounded and the key quantities \mathcal{B}, \mathcal{H} are finite. However, the bound provided by the proposition did not involve the supremum norms of U, DU but the local BMO norm of U in M) and the constants in A) and F).

In applications, the following corollary of the above theorem will be more applicable.

Corollary 3.6 *The conclusion of Theorem 3.2 holds true if U) and M) are replaced by the following condition.*

M') *There is a constant C such that for any solution U to (3.7).*

$$\|U\|_{L^1(\Omega)}, \|DU\|_{L^n(\Omega)} \leq C. \quad (3.16)$$

Proof: By Hölder's inequality it is clear that M') implies U). To establish the uniform smallness condition M) we can argue by contradiction. We only sketch the idea of the argument here. If M) is not true then there are sequences of reals $\{\sigma_n\}$ in $[0, 1]$ and $\{U_n\}$ of solutions of (3.7) converges weakly to some U in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ but $\|U_n\|_{BMO(B_{r_n})} > \varepsilon_0$ for some $\varepsilon_0 > 0$ and a positive sequence $\{r_n\}$ converging to 0. We then have $\|U_n\|_{BMO(B_R)}$ converge to $\|U\|_{BMO(B_R)}$ for any given $R > 0$. Since DU_n is uniformly bounded in $L^n(\Omega)$, it is not difficult to see that $DU \in L^n(\Omega)$. Hence, by Poincaré's inequality and the continuity of the integral of $|DU|^n$, $\|U\|_{BMO(B_R)}$ can be arbitrarily small. Clearly, if $r_n < R$ then $\|U_n\|_{BMO(B_{r_n})} \leq \|U_n\|_{BMO(B_R)}$. Choosing R sufficiently small and letting n tend to infinity, $\|U_n\|_{BMO(B_{r_n})}$ can be arbitrarily small. We obtain a contradiction. Hence, M) is true and the proof is complete. ■

Remark 3.7 Consider a family of systems (not necessarily defined as in the proof)

$$\begin{cases} -\operatorname{div}(\hat{A}_\sigma(U, DU)) = \hat{F}_\sigma(U, DU) \text{ in } \Omega, \\ U \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega, \end{cases}$$

which satisfies uniformly the assumptions A), f.0)-f.2) and SG) and that the number \mathbf{A} in (3.8) is bounded. If any strong solutions U of the family satisfies U) and M) (or M')) uniformly then argument in the proof of Theorem 3.2 shows that there is a constant C depending only on the parameters in A), f.0)-f.2), SG), U), M) and \mathbf{A} in (3.8) such that $\|U\|_{\mathbf{X}} \leq C$.

3.2 Some Examples:

We now present some examples in applications where Theorem 3.2 or Corollary 3.6 can apply. The main theme in these examples is to establish the uniform bounds (3.16) for the norms $\|\cdot\|_{L^1(\Omega)}$ and $\|D(\cdot)\|_{L^n(\Omega)}$ of solutions to (3.7). In fact, under suitable assumptions on the structural of (3.7), we will show that it is sufficient to control L^1 norms of the solutions (see Remark 3.4).

For simplicity we will consider only the following quasilinear system

$$\begin{cases} -\operatorname{div}(A(u)Du) = f(u) \text{ in } \Omega, \\ u \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega. \end{cases} \quad (3.17)$$

Following Remark 3.1, we define $\hat{f}_\sigma(u, \zeta) = f(\sigma u)$. The corresponding version of (3.7) is

$$\begin{cases} -\operatorname{div}(A(\sigma u)Du) = f(\sigma u) \text{ in } \Omega, \sigma \in [0, 1], \\ u \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega. \end{cases} \quad (3.18)$$

It is clear that (3.17) is (3.1) with $A(u, \zeta), \hat{f}(u, \zeta)$ being $A(u)\zeta, f(u)$. We will assume that these data satisfy A) and F) and that $\lambda(u), f(u)$ have comparable polynomial growths.

G) Assume that $\lambda(u) \sim (1+|u|)^k$ and $|f(u)| \leq C|u|^{l+1} + C$ for some $C, k > 0$ and $0 \leq l \leq k$.

We first have the following

Lemma 3.8 *Assume G). There is a constant C such that the following holds true for any solution u to (3.18).*

$$\int_{\Omega} (1 + |\sigma u|^k) |D(\sigma u)|^2 dx \leq C\sigma^{k+3} \|u\|_{L^1(\Omega)}^{k+2} + C\sigma^2 \|u\|_{L^1(\Omega)}. \quad (3.19)$$

Proof: Testing the system (3.18) with $\sigma^2 u$ and using the ellipticity assumption, we obtain

$$\sigma^2 \int_{\Omega} \lambda(\sigma u) |Du|^2 dx \leq C\sigma \int_{\Omega} \langle f(\sigma u), \sigma u \rangle dx.$$

From the growth assumptions on $\lambda(u), f(u)$ in G) and a simple use of Young's inequality applying to the right hand side of the above inequality, one gets

$$\int_{\Omega} (1 + |\sigma u|^k) |D(\sigma u)|^2 dx \leq C\sigma \int_{\Omega} (|\sigma u|^{k+2} + |\sigma u|) dx. \quad (3.20)$$

We now recall the following inequality, which can be proved easily by using a contradiction argument and the fact that $W^{1,2}(\Omega)$ is embedded compactly in $L^2(\Omega)$: For any $w \in W^{1,2}(\Omega)$, $\varepsilon > 0$ and $\alpha \in (0, 1]$ there exists a constant $C(\varepsilon, \alpha)$ such that

$$\int_{\Omega} |w|^2 dx \leq \varepsilon \int_{\Omega} |Dw|^2 dx + C(\varepsilon, \alpha) \left(\int_{\Omega} |w|^\alpha dx \right)^{\frac{2}{\alpha}}. \quad (3.21)$$

Setting $w = |\sigma u|^{\frac{k+2}{2}}$ and noting that $w^2 = |\sigma u|^{k+2}$ and $|Dw|^2 \sim |\sigma u|^k |D(\sigma u)|^2$. Using the above inequality for $\alpha = 2/(k+2)$ and sufficiently small $\varepsilon > 0$, we deduce from (3.20)

$$\int_{\Omega} (1 + |\sigma u|^k) |D(\sigma u)|^2 dx \leq C\sigma^{k+3} \left(\int_{\Omega} |u| dx \right)^{k+2} + C\sigma^2 \int_{\Omega} |u| dx.$$

This is (3.19) and the proof is complete. ■

In particular, (3.19) implies that

$$\int_{\Omega} |Du|^2 dx \leq C\sigma^{k+1} \|u\|_{L^1(\Omega)}^{k+2} + C\|u\|_{L^1(\Omega)}.$$

Hence, as $k > 0$ and $\sigma \in [0, 1]$, there is a constant C depending only on $\|u\|_{L^1(\Omega)}$ such that $\|Du\|_{L^2(\Omega)} \leq C$ for all solutions to (3.18). The following result immediately follows from this fact and Corollary 3.6.

Corollary 3.9 *Assume G) and that $n = 2$. If the solutions of (3.18) are uniformly bounded in $L^1(\Omega)$ then the system (3.17) has a strong solution.*

For the case $n = 3, 4$ we consider some C^2 map $P : \mathbb{R}^m \rightarrow \mathbb{R}^m$ we consider the generalized SKT system

$$\begin{cases} -\Delta(P(u)) = f(u) \text{ in } \Omega, \\ u \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega. \end{cases} \quad (3.22)$$

This system is a generalized version of the SKT model (see [25] where $m = 2, n \leq 2$ and the components of $P(u)$ are assumed to be quadratics). The above system is a special case of (3.17) if we set $A(u) = P_u(u)$ and assume A) and F).

We then have the following

Corollary 3.10 *Assume G) and that $n \leq 4$. If the solutions of (3.23) are uniformly bounded in $L^1(\Omega)$ then the system (3.22) has a strong solution.*

Proof: Since $D(P(\sigma u)) = \sigma A(\sigma u) Du$, the system (3.18) now reads

$$\begin{cases} -\Delta(P(\sigma u)) = \sigma f(\sigma u) \text{ in } \Omega, \\ u \text{ satisfies homogeneous Dirichlet or Neumann BC on } \partial\Omega. \end{cases} \quad (3.23)$$

If $n = 2$ the result was proved in Corollary 3.9. We only consider the case $n = 4$ as the case $n = 3$ is similar and simpler. Again, in this proof, let us denote $w := |\sigma u|^{\frac{k+2}{2}}$ and

$M := \|u\|_{L^1(\Omega)}$. From (3.19) we see that $w \in W^{1,2}(\Omega)$ and we can find constants $C_i(M)$ such that

$$\|w\|_{W^{1,2}(\Omega)} \leq \sigma C_1(M) \Rightarrow \|w\|_{L^4(\Omega)} \leq \sigma C_2(M), \quad (3.24)$$

using Sobolev's imbedding theorem. From the growth condition on f in G) and Young's inequality, $|f(\sigma u)| \leq C(w^2 + 1)$. Therefore, the above estimates and the equation in (3.23) imply

$$\|f(\sigma u)\|_{L^2(\Omega)} \leq C_3(M) \Rightarrow \|\Delta(P(\sigma u))\|_{L^2(\Omega)} \leq \sigma C_4(M). \quad (3.25)$$

On the other hand, since $|D(P(\sigma u))| \sim (1 + |\sigma u|^k)|D(\sigma u)|$, we can use Hölder's inequality and (3.19) the bound for $\|f(\sigma u)\|_{L^2(\Omega)}$ to see that

$$\int_{\Omega} |D(P(\sigma u))| \, dx \leq \|(1 + |\sigma u|^k)\|_{L^2(\Omega)} \|D(\sigma u)\|_{L^2(\Omega)} \leq \sigma C_5(M).$$

Thus, $D(P(\sigma u)) \in L^1(\Omega)$. The last inequality in (3.25) and Schauder's estimates imply $\|D^2(P(\sigma u))\|_{L^2(\Omega)} \leq \sigma C_6(M)$ for some constant $C_6(M)$. By Sobolev's inequality,

$$\|D(P(\sigma u))\|_{L^4(\Omega)} \leq \sigma C_7(M).$$

Because $A(u) = P_u(u)$ and $D(P(\sigma u)) = \sigma A(\sigma u)Du$, we have $Du = \sigma^{-1}A^{-1}(\sigma u)D(P(\sigma u))$. As $A(u)$ is elliptic, its inverse is bounded by some constant C . We derive from these facts and the above estimate that

$$\|Du\|_{L^4(\Omega)} \leq \sigma^{-1}C\|D(P(\sigma u))\|_{L^4(\Omega)} \leq CC_7(M).$$

This gives a uniform estimate for $\|Du\|_{L^4(\Omega)}$ and completes the proof. ■

We end this section by discussing some special cases where the L^1 norm can actually be controlled uniformly so that the above corollaries are applicable.

Inspired by the SKT model in [25] with competitive Lotka-Volterra reactions, we consider the following situation.

SKT) For some $k > 0$ assume that $\lambda(u) \sim (1 + |u|)^k$ and $f_i(u) = u_i(d_i - g_i(u))$ for some C^1 function $g(u) = (g_1(u), \dots, g_m(u))$ satisfying

$$|g(u)|, |u||\partial_u g(u)| \leq C|u|^k \quad (3.26)$$

for some positive constant C .

Corollary 3.11 *Assume SKT). Suppose that there is a positive constants C_1 such that*

$$\sum_i \langle u_i g_i(u), u_i \rangle \geq C_1 |u|^{k+2}. \quad (3.27)$$

Then there is a strong solution to (3.17) (resp. (3.22) when $n = 2$ (resp. $n \leq 4$).

In addition, if Neumann boundary condition is assumed then (3.27) can be replaced by

$$\sum_i g_i(u)u_i \geq C_1 |u|^{k+1}. \quad (3.28)$$

Proof: We now replace $f(\sigma u)$ in (3.18) by $f_\sigma(u) = (f_{1,\sigma}(u), \dots, f_{m,\sigma}(u))$ with

$$f_{i,\sigma}(u) = \sigma^k d_i u_i - u_i g_i(\sigma u).$$

Since $|Df_\sigma(u)| \leq C[\sigma^\tau + |g(\sigma u)| + \sigma|u||\partial_{\sigma u} g(\sigma u)||Du|]$, we see that f_σ will satisfy f.0)-f.2) if the growth condition (3.26) holds. It is easy to see that the argument in the proof of Lemma 3.8 continues to hold with this new choice of f_σ and gives (3.19). Hence, the assertions on existence of strong solutions of the above corollaries continues to hold if we can uniformly control the $L^1(\Omega)$ norm of the solutions. This is exactly what we will do in the sequel.

Let us consider the assumption (3.27) first. We deduce from (3.27) that $\langle u_i g_i(\sigma u), u_i \rangle \geq C_1 \sigma^k |u|^{k+2}$. Therefore, testing the system (3.7) with u , we obtain

$$\int_{\Omega} \lambda(\sigma u) |Du|^2 dx \leq C_1 \sigma^k \int_{\Omega} |u|^2 dx - C_2 \sigma^k \int_{\Omega} |u|^{k+2} dx.$$

Let $w = u$ in (3.21) and multiply the result with σ^k to have

$$\sigma^k \int_{\Omega} |u|^2 dx \leq \varepsilon \sigma^k \int_{\Omega} |Du|^2 dx + C(\varepsilon, \alpha) \sigma^k \left(\int_{\Omega} |u| dx \right)^2.$$

Because $\lambda(\sigma u) \geq \lambda_0 > 0$, for sufficiently small ε we deduce from the above two inequalities that there is a constant C_4 such that

$$C_2 \sigma^k \int_{\Omega} |u|^{k+2} dx \leq C_4 \sigma^k \left(\int_{\Omega} |u| dx \right)^2.$$

Applying Hölder's inequality to the left hand side integral, we derive

$$C_5 \left(\int_{\Omega} |u| dx \right)^{k+2} \leq C_4 \left(\int_{\Omega} |u| dx \right)^2, \quad C_5 > 0.$$

Since $k > 0$, the above inequality shows that $\|u\|_{L^1(\Omega)}$ is bounded by a fixed constant.

We now consider the assumption (3.28) and assume the Neumann boundary condition. Testing the system with 1, we obtain

$$\int_{\Omega} g_i(\sigma u) u_i dx = \sigma^k \int_{\Omega} d_i u_i dx.$$

From (3.28), $\sum_i g_i(\sigma u) u_i \geq C_1 \sigma^k |u|^{k+1}$. We then derive from the above equation the following

$$C_1 \sigma^k \int_{\Omega} |u|^{k+1} dx \leq C(d_i) \sigma^k \int_{\Omega} |u| dx.$$

Again, applying Hölder's inequality to the left hand side integral, we derive

$$C_2 \left(\int_{\Omega} |u| dx \right)^{k+1} \leq C(d_i) \int_{\Omega} |u| dx$$

for some positive constant C_2 . Again, as $k > 0$, the above gives the desired uniform estimate for $\|u\|_{L^1(\Omega)}$. The proof is complete. ■

Remark 3.12 The conditions (3.27) and (3.28) on the positive definiteness of g need only be assumed for u such that $|u| \geq M$ for some positive M .

4 On Trivial and Semi Trivial Solutions

We now see that Theorem 3.2 establishes the existence of a strong solution in \mathbf{X} to (3.1). However, the conclusion of this theorem does not provide useful information if some 'trivial' or 'semi trivial' solutions, which are solutions to a subsystem of (3.1), are obviously guaranteed by other means. We will be interested in finding other nontrivial solutions to (3.1). To this end, we will first investigate these 'trivial' or 'semi trivial' solutions. Several sufficient conditions for nontrivial solutions to exist will be presented in Section 4.2.

Many results in this section, in particular the abstract results in Section 4.1, can apply to the general (3.1). However, for simplicity of our presentation we restrict ourselves to the system

$$\begin{cases} -\operatorname{div}(A(u)Du) = \hat{f}(u) & \text{in } \Omega, \\ \text{Homogenous Dirichlet or Neumann boundary conditions} & \text{on } \partial\Omega. \end{cases} \quad (4.1)$$

As in the previous section, we fix some $\alpha_0 > 0$ and let \mathbf{X} be $C^{1,\alpha_0}(\Omega, \mathbb{R}^m)$ (or $C^{1,\alpha_0}(\Omega) \cap C_0(\Omega)$ if Dirichlet boundary conditions are considered). Under appropriate assumptions, Theorem 3.2 gives the existence of a strong solution in \mathbf{X} to (4.1). This solution may be trivial. For examples, the *trivial solution* $u = 0$ is a solution to the system if $\hat{f}(0) = 0$.

Let us discuss the existence of *semi trivial* solutions. We write $\mathbb{R}^m = \mathbb{R}^{m_1} \oplus \mathbb{R}^{m_2}$ for some $m_1, m_2 \geq 0$ and denote $\mathbf{X}_i = C^{1,\alpha_0}(\Omega, \mathbb{R}^{m_i})$. By reordering the equations and variables, we write $\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2$, an element of \mathbf{X} as (u, v) with $u \in \mathbf{X}_1$, $v \in \mathbf{X}_2$, and

$$A(u, v) = \begin{bmatrix} P^{(u)}(u, v) & P^{(v)}(u, v) \\ Q^{(u)}(u, v) & Q^{(v)}(u, v) \end{bmatrix} \text{ and } \hat{f}(u, v) = \begin{bmatrix} f^{(u)}(u, v) \\ f^{(v)}(u, v) \end{bmatrix}.$$

Here, $P^{(u)}(u, v)$ and $Q^{(v)}(u, v)$ are matrices of sizes $m_1 \times m_1$ and $m_2 \times m_2$ respectively.

Suppose that

$$Q^{(u)}(u, 0) = 0 \text{ and } f^{(v)}(u, 0) = 0 \quad \forall u \in \mathbf{X}_1, \quad (4.2)$$

then $(u, 0)$, with $u \neq 0$, is a semi trivial solution if u solves the subsystem

$$-\operatorname{div}(P^{(u)}(u, 0)Du) = f^{(u)}(u, 0).$$

For each $u \in \mathbf{X}$ and some constant matrix K we consider the following *linear* elliptic system for w .

$$\begin{cases} -\operatorname{div}(A(u)Dw) + Kw = \hat{f}(u) + Ku & \text{in } \Omega, \\ \text{Homogenous boundary conditions} & \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

For a suitable choice of K , see (3.3), we can always assume that (4.3) has a unique weak solution $w \in \mathbf{X}$. This is equivalent to say that the elliptic system

$$\begin{cases} -\operatorname{div}(A(u)Dw) + Kw = 0 & x \in \Omega, \\ \text{Homogenous boundary conditions} & \text{on } \partial\Omega \end{cases} \quad (4.4)$$

has $w = 0$ as the only solution. This is the case if we assume that there is $k > 0$ such that

$$\langle Ku, u \rangle \geq k|u|^2 \quad \forall u \in \mathbb{R}^m.$$

We then define $T(u) := w$ with w being the weak solution to (4.3). It is clear that the fixed point solutions of $T(u) = u$ are solutions to (4.1), where $w = u$.

Since $A(u)$ is C^1 in u , $A(u(x))$ is Hölder continuous on Ω . The regularity theory of linear elliptic systems then shows that $w \in C^{1,\alpha}(\Omega, \mathbb{R}^m)$ for all $\alpha \in (0, 1)$ so that T is compact in \mathbf{X} . Furthermore, T is a differentiable map.

If (4.1) satisfies the assumptions of Theorem 3.2 then there is $M > 0$ such that

$$T(u) = u \Rightarrow \|u\|_{\mathbf{X}} < M. \quad (4.5)$$

In applications, we are also interested in finding solutions that are positive. We then consider the positive cone in \mathbf{X}

$$\mathbf{P} := \{u \in \mathbf{X} : u = (u_1, \dots, u_m), u_i \geq 0 \forall i\},$$

which has nonempty interior

$$\dot{\mathbf{P}} := \{u \in \mathbf{X} : u = (u_1, \dots, u_m), u_i > 0 \forall i\}.$$

Let M be the number provided by Theorem 3.2 in (4.5). We denote by $\mathbf{B} := B_{\mathbf{X}}(0, M)$ the ball in \mathbf{X} centered at 0 with radius M . If T maps $\mathbf{B} \cap \mathbf{P}$ into \mathbf{P} then, since \mathbf{P} is closed in \mathbf{X} and convex and it is a retract of \mathbf{X} (see [4]), we can define the cone index $\text{ind}(T, U, \mathbf{P})$ for any open subset U of $\mathbf{B} \cap \mathbf{P}$ as long as T has no fixed point on ∂U , the boundary of U in \mathbf{P} ([1, Theorem 11.1]).

The argument in the proof of Theorem 3.2 can apply here to give

$$\text{ind}(T, \mathbf{B} \cap \mathbf{P}, \mathbf{P}) = 1.$$

This yields the existence of a fixed point of T , or a solution to (4.1), in \mathbf{P} . From the previous discussion, this solution may be trivial or semi trivial. To establish the existence of a nontrivial positive solution u , i.e. $u \in \dot{\mathbf{P}}$, we will compute the local indices of T at its trivial and semi trivial fixed points. If these indices do not add up to $\text{ind}(T, \mathbf{B} \cap \mathbf{P}, \mathbf{P}) = 1$ then the existence of nontrivial solutions follows from [1, Corollary 11.2].

4.1 Some general index results

We then consider the case when (4.1) has trivial or semi trivial solutions. That is when $u = 0$ or some components of u is zero. We will compute the local indices of the map $T(u)$ at these trivial or semi trivial solutions. The abstract results in this section are in fact independent of (4.1) and thus can apply to (3.1) and other general situations as well.

We decompose \mathbf{X} as $\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2$ and denote by \mathbf{P}_i and $\dot{\mathbf{P}}_i$, $i = 1, 2$, the positive cones and their nonempty interiors in \mathbf{X}_i 's. We assume (see also (4.5)) that there is $M > 0$ such that the map T is well defined as a map from the ball \mathbf{B} centered at 0 with radius M into \mathbf{P} . Accordingly, we denote $\mathbf{B}_i = \mathbf{B} \cap \mathbf{X}_i$.

For $(u, v) \in \mathbf{B}_1 \oplus \mathbf{B}_2$, we write

$$T(u, v) = (F_1(u, v), F_2(u, v)),$$

where F_i 's are maps from \mathbf{B} into \mathbf{X}_i . We also write $\partial_u F_i, \partial_v F_i, \dots$ for the partial Fréchet derivatives of these maps.

It is clear that for $\phi = (\phi_1, \phi_2) \in \mathbf{X}_1 \oplus \mathbf{X}_2$

$$T'(u, v)\phi = (\partial_u F_1(u, v)\phi_1 + \partial_v F_1(u, v)\phi_2, \partial_u F_2(u, v)\phi_1 + \partial_v F_2(u, v)\phi_2).$$

For any fixed $u \in \mathbf{B}_1$ and $v \in \mathbf{B}_2$, we will think of $F_1(\cdot, v)$ and $F_2(u, \cdot)$ as maps from \mathbf{B}_1 into \mathbf{X}_1 and from \mathbf{B}_2 into \mathbf{X}_2 respectively. With a slight abuse of the notation, we still write $\partial_u F_1, \partial_v F_2$ for the Fréchet derivatives of these maps.

Taking into account of (4.2), we will therefore assume in the sequel that

$$F_2(u, 0) = 0 \quad \forall u \in \mathbf{P}_1. \quad (4.6)$$

This implies

$$F_2(u, tv) = t \int_0^1 \partial_v F_2(u, tsv) v \, ds, \quad (4.7)$$

where $\partial_v F_2(u, \cdot)$ is the derivative of $F_2(u, \cdot) : \mathbf{B}_2 \rightarrow \mathbf{X}_2$.

Let Z_1 be the set of fixed points of $F_1(\cdot, 0)$ in \mathbf{P}_1 and assume that $Z_1 \neq \emptyset$. Of course, $u \in Z_1$ iff $F_1(u, 0) = u$ and $F_2(u, 0) = 0$.

For each $u \in \mathbf{B}_1$ we consider the spectral radius $r_v(u)$ of $\partial_v F_2(u, 0)$.

$$r_v(u) = \lim_{k \rightarrow \infty} \|\partial_v F_2(u, 0)\|_{L(\mathbf{X}_2)}^{1/k}.$$

We also consider the following subsets of Z_1

$$Z_1^+ = \{u \in Z_1 : r_v(u) > 1\}, \quad Z_1^- = \{u \in Z_1 : r_v(u) < 1\}. \quad (4.8)$$

Roughly speaking, Z_1^+ (resp. Z_1^-) consists of unstable (resp. stable) fixed points of T in the \mathbf{P}_2 -direction. Sometimes we simply say that an element in Z_1^+ (resp. Z_1^-) is *v-unstable* (resp. *v-stable*).

Let us fix an open neighborhood U of Z_1 in \mathbf{P}_1 . We first need to show that the index $\text{ind}(T, U \oplus V)$ is well defined for some appropriate neighborhood of V in \mathbf{P}_2 , i.e. $U \oplus V$ is a neighborhood of Z_1 as a subset of \mathbf{P} and T has no fixed point on its boundary. To this end, we will always assume that

Z) If $u \in Z_1$ then $\partial_v F_2(u, 0)$, the Frechet derivative of $F_2(u, \cdot) : \mathbf{B}_2 \rightarrow \mathbf{X}_2$, does not have a positive eigenvector to the eigenvalue 1.

In what follows, if G is a map from an open subset W of \mathbf{P}_i into \mathbf{P}_i and there is no ambiguity can arise then we will abbreviate $\text{ind}(G, W, \mathbf{P}_i)$ by $\text{ind}(G, W)$. We also say that G is a **strongly** positive endomorphism on W into \mathbf{X}_i if G maps $W \cap \dot{\mathbf{P}}_i$ into $\dot{\mathbf{P}}_i$.

The following main result of this section shows that $\text{ind}(T, U \oplus V)$ is determined by the index of the restriction $T|_{\mathbf{X}_1}$, i.e. $F_1(\cdot, 0)$, at *v-stable* fixed points (in Z_1^-).

Theorem 4.1 *Assume Z). There is a neighborhood of V of 0 in \mathbf{P}_2 such that $\text{ind}(T, U \oplus V)$ is well defined.*

Suppose also the following.

- i) T is a positive map. That is, T maps $\mathbf{B} \cap \mathbf{P}$ into \mathbf{P} .
- ii) $F_2(u, 0) = 0$ for all $u \in \mathbf{B}_1$.
- iii) At each semi trivial fixed point $u \in Z_1$, $\partial_v F_2(u, 0)$ is a **strongly** positive map on \mathbf{B}_2 into \mathbf{X}_2 .

Then there exist two disjoint open sets U^+, U^- in U such that $Z_1^+ \subset U^+$ and $Z_1^- \subset U^-$ and

$$\text{ind}(T, U \oplus V) = \text{ind}(T, U^- \oplus V) = \text{ind}(F_1(\cdot, 0), U^-).$$

Remark 4.2 For $u \in Z_1$, i) implies that $\partial_v F_2(u, 0)$ is a positive endomorphism on \mathbf{X}_2 . In fact, for any $u \in Z_1$, $x > 0$ and positive small t such that $tx \in V$ we have by our assumptions that $F_2(u, tx) \geq 0$ and $F_2(u, 0) = 0$. Hence, $\partial_v F_2(u, 0)x = \lim_{t \rightarrow 0^+} t^{-1} F_2(u, tx) \geq 0$. So that $\partial_v F_2(u, 0)$ is positive. If certain strong maximum principle for the linear elliptic system defining $\partial_v F_2(u, 0)$ is available, see [1, Theorem 4.2], then $\partial_v F_2(u, 0)$ is *strongly* positive and iii) follows. This assumption can be relaxed if Z_1 is a singleton (see Remark 4.7).

Remark 4.3 If $\partial_v F_2(u, 0)$ is strongly positive then $r_v(u)$ is the only eigenvalue with positive eigenfunction. Therefore, the assumptions $r_v(u) < 1$ and $r_v(u) > 1$ are respectively equivalent to the followings

- I'.1)** $\partial_v F_2(u, 0)$ does not have any positive eigenvector to any eigenvalue $\lambda > 1$.
- I'.2)** $\partial_v F_2(u, 0)$ has a positive eigenvector to some eigenvalue $\lambda > 1$.

The proof of Theorem 4.1 will be divided into several lemmas which can be of interest in themselves.

Our first lemma shows that there exists a neighborhood V claimed in Theorem 4.1 such that $\text{ind}(T, U \oplus V)$ is well defined.

Lemma 4.4 Assume Z). There is $r > 0$ such that for $V = B(0, r) \cap \mathbf{P}_2$, the ball in \mathbf{P}_2 centered at 0 with radius $r > 0$, there is no fixed point of $T(u, v) = (u, v)$ with $v > 0$ in the closure of $U \oplus V$ in \mathbf{P} .

Proof: By contradiction, suppose that there are sequences $\{r_n\}$ of positives $r_n \rightarrow 0$ and $\{u_n\} \subset U$, $\{v_n\} \subset \mathbf{P}_2$ with $\|v_n\| = r_n$ such that, using (4.7)

$$u_n = F_1(u_n, v_n), v_n = F_2(u_n, v_n) = \int_0^1 \partial_v F_2(u_n, sv_n) v_n ds.$$

Setting $w_n = v_n / \|v_n\|$ we have

$$w_n = \int_0^1 \partial_v F_2(u_n, sr_n w_n) w_n ds.$$

By compactness, via a subsequence of $\{u_n\}$, and continuity we can let $n \rightarrow \infty$ and obtain $u_n \rightarrow u$ for some $u \in Z_1$, $v_n \rightarrow 0$ and $w_n \rightarrow w$ in \mathbf{X}_2 such that $u = F(u, 0)$ and $\|w\| = 1$. Hence, $w > 0$ and satisfies

$$w = \int_0^1 \partial_v F_2(u, 0) w ds = \partial_v F_2(u, 0) w.$$

Thus, w is a positive eigenvector of $\partial_v F_2(u, 0)$ to the eigenvalue 1. This is a contradiction to Z) and completes the proof. ■

In the sequel, we will always denote by V the neighborhood of 0 in \mathbf{P}_2 as in the above lemma.

Our next lemma on the index of T shows that T can be computed by using its restriction and partial derivatives.

Lemma 4.5 *We have*

$$\text{ind}(T, U \oplus V) = \text{ind}(T_*, U \oplus V),$$

where $T_*(u, v) = (F_1(u, 0), \partial_v F_2(u, 0)v)$.

Proof: Consider the following homotopy

$$H(t, u, v) = \left(F_1(u, tv), \int_0^1 \partial_v F_2(u, tsv) v \, ds \right) \text{ for } t \in [0, 1]. \quad (4.9)$$

We show that this homotopy is well defined on $U \oplus V$. Indeed, if $H(t, u, v)$ has a fixed point (u, v) on the boundary of $U \oplus V$ for some $t \in [0, 1]$ then

$$F_1(u, tv) = u, \quad \int_0^1 \partial_v F_2(u, tsv) v \, ds = v, \quad (u, v) \in \partial(U \oplus V).$$

Assume first that $t > 0$. If $v = 0$ then the first equation gives that $F_1(u, 0) = u$ so that $u \in Z_1$. But then $(u, 0) \notin \partial(U \oplus V)$. Thus, $v > 0$ and the second equation (see (4.7)) yields $F_2(u, tv) = tv$. This means (u, tv) is a fixed point of T in the closure of $U \oplus V$ with $tv > 0$. But there is no such fixed point of $T(u, v) = (u, v)$ in the closure of $U \oplus V$ by Lemma 4.4. Hence, $H(t, u, v)$ cannot have a fixed point (u, v) on the boundary of $U \oplus V$ if $t > 0$. We then consider $H(0, u, v)$ whose fixed points $(u, v) \in \partial(U \oplus V)$ must satisfy $u = F_1(u, 0)$ so that $u \in Z_1$ and $\partial_v F_2(u, 0)v = v$ with $v > 0$. But this contradicts Z).

Thus the homotopy is well defined and we have that

$$\text{ind}(T, U \oplus V) = \text{ind}(H(1, \cdot), U \oplus V) = \text{ind}(H(0, \cdot), U \oplus V).$$

By (4.9), $H(0, u, v) = (F_1(u, 0), \partial_v F_2(u, 0)v) = T_*(u, v)$. The proof is complete. ■

We now compute $\text{ind}(T_*, U \oplus V)$.

Lemma 4.6 *Assume that $\partial_v F_2(u, 0)$ is a strongly positive endomorphism on \mathbf{B}_2 into \mathbf{X}_2 for each $u \in Z_1$. The following holds*

I.1) *If $r_v(u) < 1$ for any $u \in Z_1$ then $\text{ind}(T_*, U \oplus V) = \text{ind}(F_1(\cdot, 0), U)$.*

I.2) *If $r_v(u) > 1$ for any $u \in Z_1$ then $\text{ind}(T_*, U \oplus V) = 0$.*

Proof: First of all, we see that $\partial_v F_2(u, 0)$ is a compact map. In fact, we have $F_2(u, 0) = 0$ so that $\partial_v F_2(u, 0)x = \lim_{t \rightarrow 0^+} t^{-1} F_2(u, tx)$. Since F_2 is compact, so is $\partial_v F_2(u, 0)$.

To prove I.1), we consider the following homotopy

$$H(u, v, t) = (F_1(u, 0), t\partial_v F_2(u, 0)v), \quad t \in [0, 1].$$

This homotopy is well defined on $U \oplus V$. Indeed, a fixed point of (u, v) of $H(\cdot, \cdot, t)$ on $\partial(U \oplus V)$ must satisfies $u \in Z_1$ and $tv > 0$. But this means $v > 0$ is a positive eigenfunction to the eigenvalue $t^{-1} \geq 1$. This is a contradiction to Z) and the Krein-Ruthman theorem, see [1, Theorem 3.2, ii)] for strongly positive compact endomorphism on \mathbf{X}_2 , $\partial_v F_2(u, 0)$ has no positive eigenvector different from $r_v(u)$, which is asumed to be less than 1 in this case. Thus,

$$\text{ind}(T_*, U \oplus V) = \text{ind}(H(\cdot, \cdot, 0), U \oplus V).$$

But $H(u, v, 0) = (F_1(u, 0), 0)$ so that, by index product theorem, $\text{ind}(H(\cdot, \cdot, 0), U \oplus V) = \text{ind}(F_1(\cdot, 0), U)$. Hence, $\text{ind}(T_*, U \oplus V) = \text{ind}(F_1(\cdot, 0), U)$.

We now consider I.2). Let h be any element in $\dot{\mathbf{P}}_2$, the interior of \mathbf{P}_2 . We first consider the following homotopy

$$H(u, v, t) = (F_1(u, 0), t\partial_v F_2(u, 0)v + th), \quad t \geq 1. \quad (4.10)$$

If $H(\cdot, t)$ has a fixed point (u, v) in $\partial(U \oplus V)$ then $u \in Z_1$ and $v > 0$. Thus, there is some $v_* > 0$ such that $v_* = t\partial_v F_2(u, 0)v_* + th$. This means $t^{-1}v_* - \partial_v F_2(u, 0)v_* = h$. Since $t^{-1} \leq 1 < r_v(u)$, this contradicts the following consequence of the Krein-Rutman theorem (see [1, Theorem 3.2, iv)] for strongly positive compact operators:

$$\lambda x - \partial_v F_2(u, 0)x = h \text{ has no positive solution if } \lambda \leq r_v(u).$$

Thus the homotopy is well defined on $U \oplus V$. Because $\partial_v F_2(u, 0)v_* \geq 0$, $t\partial_v F_2(u, 0)v_* + th$ becomes unbounded as $t \rightarrow \infty$, it is clear that $H(u, v, t)$ has no fixed point in $U \oplus V$ for t large. We then have $\text{ind}(H(\cdot, \cdot, 1), U \oplus V) = 0$.

We now consider the homotopy

$$G(u, v, t) = (F_1(u, 0), \partial_v F_2(u, 0)v + th) \quad t \in [0, 1].$$

We will see that this homotopy is well defined on $U \oplus V$ if $\|h\|_{\mathbf{X}_2}$ is sufficiently small. First of all, since $\partial_v F_2(u, 0)$ is a compact map, the map $f(v) = v - \partial_v F_2(u, 0)v$ is a closed map so that $f(\partial V)$ is closed. By Z), if $u \in Z_1$ then $0 \notin f(\partial V)$ so that there is $\varepsilon > 0$ such that $B_\varepsilon(0) \cap f(\partial V) = \emptyset$. This means

$$\|v - \partial_v F_2(u, 0)v\|_{\mathbf{X}_2} > \varepsilon \quad \forall v \in \partial V. \quad (4.11)$$

We now take h such that $\|h\|_{\mathbf{X}_2} < \varepsilon/2$. If $G(\cdot, \cdot, t)$ has a fixed point $(u, v) \in \partial(U \oplus V)$ then $u \in Z_1$, $v \in \partial V$ and $v - \partial_v F_2(u, 0)v = th$. This fact and (4.11) then yield

$$\|v - \partial_v F_2(u, 0)v\|_{\mathbf{X}_2} > \varepsilon > \|th\|_{\mathbf{X}_2} \quad \forall t \in [0, 1].$$

This means $v - \partial_v F_2(u, 0)v \neq th$ for all $u \in Z_1$, $v \in \partial V$. Hence, the homotopy defined by G is well defined on $U \oplus V$. We then have

$$\text{ind}(T_*, U \oplus V) = \text{ind}(G(\cdot, \cdot, 0), U \oplus V) = \text{ind}(H(\cdot, \cdot, 1), U \oplus V) = 0.$$

The proof is complete. ■

Remark 4.7 If we drop the assumption that $\partial_v F_2(u, 0)$ is strongly positive then the conclusion of Lemma 4.6 continues to hold if I.1) is replaced by I'.1), which is essentially used in the argument. This is also the case, if we assume I'.2) in place of I.2) and Z_1 a singleton, $Z_1 = \{u\}$. In fact, let h be such a positive eigenvector of $\partial_v F_2(u, 0)$ for some $\lambda_u > 1$. We consider the following homotopy.

$$H(u, v, t) = (F_1(u, 0), \partial_v F_2(u, 0)v + th), \quad t \geq 0.$$

We will show that the homotopy is well defined. The case $t = 0$ is easy. Indeed, if $H(\cdot, \cdot, 0)$ has a fixed point (u, v) in $\partial(U \oplus V)$ then $u \in Z_1$ and $\partial_v F_2(u, 0)v = v$. But this gives $v = 0$, by Z), and u is not in ∂U .

We consider the case $t > 0$. If $H(\cdot, t)$ has a fix point (u, v) in $\partial(U \oplus V)$ then $u \in Z_1$ and $v > 0$. Thus, there is some $v_* > 0$ such that $v_* = \partial_v F_2(u, 0)v_* + th$. Let τ_0 be the maximal number such that $v_* > \tau_0 h$. We then have $\partial_v F_2(u, 0)v_* \geq \partial_v F_2(u, 0)\tau_0 h$ so that (as $\lambda > 1$)

$$v_* = \partial_v F_2(u, 0)v_* + th \geq \partial_v F_2(u, 0)\tau_0 h + th = (\lambda\tau_0 + t)h > (\tau_0 + t)h.$$

Since $t > 0$, the above contradicts the maximality of τ_0 . Thus the homotopy is well defined. Again, when t is sufficiently large $H(u, v, t)$ has no solution in $U \oplus V$. Therefore, $\text{ind}(H(\cdot, \cdot, 1), U \oplus V) = 0$.

Proof of Theorem 4.1: The assumption i) and the regularity results in the previous section show that T is a compact map on \mathbf{X} so that $\text{ind}(T, O, \mathbf{P})$ is well defined whenever T has no fixed point on the boundary of an open set O in \mathbf{P} . The assumptions ii) and iii) allow us to make use of the lemmas in this section.

We first prove that $r_v(u)$ is continuous in $u \in Z_1$ (see also Remark 4.8). Let $\{u_n\} \subset Z_1$ be a sequence converging to some $u_* \in Z_1$. Accordingly, let h_n be the normalized eigenfunction ($\|h_n\| = 1$) to the eigenvalue $\lambda_n = r_v(u_n)$. Because $\|\partial_v F_2(u_n, 0)\|_{L(\mathbf{X}_2)}$ is bounded for all n , we see that $\{\lambda_n\}$ is bounded from the definition of the spectral radius. Let $\{\lambda_{n_k}\}$ be a convergent subsequence of $\{\lambda_n\}$ converges to some λ . The regularity of elliptic systems yields that the corresponding eigenfunction sequence $\{h_{n_k}\}$ has a convergent subsequence converges to a solution $h > 0$ of the eigenvalue problem $\partial_v F_2(u_*, 0)h = \lambda h$. By uniqueness of the positive eigenfunction (see [1, Theorem 3.2, ii)]), $\lambda = r_v(u_*)$. We now see that all convergent subsequences of $\{\lambda_n\}$ converge to $r_v(u_*)$. Thus, $\limsup \lambda_n = \liminf \lambda_n$ and $\lambda_n = r_v(u_n) \rightarrow r_v(u_*)$ as $n \rightarrow \infty$. Hence, $r_v(u)$ is continuous in $u \in Z_1$.

Therefore, Z_1^+, Z_1^- are disjoint open sets in Z_1 . By Z), their union is the compact set Z_1 so that they are also closed in Z_1 and compact in \mathbf{X}_1 . Hence, there are disjoint open sets U^+, U^- in \mathbf{X}_1 such that $Z_1^+ \subset U^+, Z_1^- \subset U^-$. We then have

$$\text{ind}(T_*, U \oplus V) = \text{ind}(T_*, (U^+ \cup U^-) \oplus V) = \text{ind}(T_*, U^+ \oplus V) + \text{ind}(T_*, U^- \oplus V).$$

Applying case I.2) of Lemma 4.6 for $U = U^+$, we see that $\text{ind}(T_*, U^+ \oplus V)$ is zero. It follows that

$$\text{ind}(T_*, U \oplus V) = \text{ind}(T_*, U^- \oplus V) = \text{ind}(F_1(\cdot, 0), U^-).$$

By Lemma 4.5, $\text{ind}(T, U \oplus V) = \text{ind}(T_*, U \oplus V)$, the theorem then follows. ■

Remark 4.8 The strong positiveness of $\partial_v F_2$ is essential in several places of our proof. Under this assumption, we provided a simple proof of the continuity of $r_v(u)$ on Z_1 . In general, as $\partial_v F_2$ is always compact, the continuity of $r_v(u)$ follows from [3, Theorem 2.1], where it was proved that the spectral radius is continuous on the subspace of compact operators.

We end this section by the following well known result which is a special case of Theorem 4.1.

Corollary 4.9 *Let X be a Banach space with positive cone P and F is a postive compact map on P . Suppose that $F(0) = 0$ and the directional Frecét derivative $F'_+(0)$ exists (i.e. $F'_+(0)x = \lim_{t \rightarrow 0^+} t^{-1}F(tx)$). Assume also that $F'_+(0)$ does not have any positive eigenvector to the eigenvalue 1 and that the following holds.*

$$F'_+(0) \text{ does not have any positive eigenvector to any eigenvalue } \lambda > 1. \quad (4.12)$$

Then we can find a neighborhood V of 0 in P such that 0 is the only fixed point of F in V and

$$\text{ind}(F, V) = \begin{cases} 1 & \text{if (4.12) holds,} \\ 0 & \text{otherwise.} \end{cases} \quad (4.13)$$

To see this, we let $\mathbf{X} = \{0\} \oplus X$, i.e. $\mathbf{X}_1 = \{0\}$ and $\mathbf{X}_2 = X$, and $T(\cdot) = (0, F(\cdot))$. Obviously, Theorem 4.1, with F_1 is the constant map and $F_2 = F$, Z_1 being the singleton $\{0\}$ (see Remark 4.7) and $U = \{0\}$, provides a neighborhood U^- of Z_1^- in \mathbf{X}_1 such that $\text{ind}(T, \{0\} \oplus V) = \text{ind}(0, U^-)$. Clearly, as F_1 is a constant map, if (4.12) holds then $U^- = \{0\}$ and $\text{ind}(F_1(\cdot, 0), U^-) = 1$; otherwise $U^- = \emptyset$ and $\text{ind}(F_1(\cdot, 0), U^-) = 0$. By the product theorem of indices, $\text{ind}(T, \{0\} \oplus V) = \text{ind}(F, V)$, (4.13) then follows.

4.2 Applications

In this section, we will show that the abstract results on the local indices of T at trivial and semi trivial solutions in Theorem 4.1 can apply to the map T defined by (4.1) satisfying a suitable set of assumptions.

Going back to the definition of T , for each $(u, v) \in \mathbf{X}$ and some suitable constant matrix K we consider the following *linear* elliptic system for $w = T(u, v)$.

$$\begin{cases} -\text{div}(A(u, v)Dw) + Kw = \hat{f}(u, v) + K(u, v) & \text{in } \Omega, \\ \text{Homogenous boundary conditions for } w & \text{on } \partial\Omega. \end{cases} \quad (4.14)$$

Remark 4.10 We observe that the choice of the matrix K is not important here as long as the map T is well defined (as a positive map). In fact, let K_1, K_2 be two different matrices and T_1, T_2 be the corresponding maps defined by (4.14). It is clear that these maps have the same set of fixed points consisting of solutions to (4.1). Hence, via a simple homotopy $tT_1 + (1 - t)T_2$ for $t \in [0, 1]$, the indices $\text{ind}(T_i, U)$ are equal whenever one of their indices is defined (i.e. (4.1) does not have any solution on ∂U).

Trivial solution: It is clear that 0 is a solution if $\hat{f}(0) = 0$. In this case, we can apply Corollary 4.9 with $F = T$. The eigenvalue problem of $T'(0)h = \lambda h$ now is

$$-\operatorname{div}(A(0)Dh) + Kh = \lambda^{-1}(\hat{f}_u(0) + K)h. \quad (4.15)$$

We then have the following result from Corollary 4.9.

Lemma 4.11 *There is a neighborhood V_0 of 0 in \mathbf{P} such that if (4.15) has a positive solution h to some eigenvalue $\lambda > 1$ then $\operatorname{ind}(T, V_0) = 0$. Otherwise, $\operatorname{ind}(T, V_0) = 1$.*

Semitrivial solution: By reordering the equations and variables, we will write an element of \mathbf{X} as (u, v) and

$$A(u, v) = \begin{bmatrix} P^{(u)}(u, v) & P^{(v)}(u, v) \\ Q^{(u)}(u, v) & Q^{(v)}(u, v) \end{bmatrix} \text{ and } \hat{f}(u, v) = \begin{bmatrix} f^{(u)}(u, v) \\ f^{(v)}(u, v) \end{bmatrix}.$$

The existence of semitrivial solutions $(u, 0)$ usually comes from the assumption that

$$Q^{(u)}(u, 0) = 0 \text{ and } f^{(v)}(u, 0) = 0 \quad \forall u \in \mathbf{X}_1. \quad (4.16)$$

If (4.16) holds then it is clear that $(u, 0)$ is a solution of (4.1) if and only if u solves the following subsystem

$$-\operatorname{div}(P^{(u)}(u, 0)Du) = f^{(u)}(u, 0). \quad (4.17)$$

Let us then assume that the set Z_1 of positive solutions to (4.17) is nonempty.

To compute the local index of T at a semi trivial solution we consider the following matrix

$$K = \begin{bmatrix} K_1^{(u)} & K_1^{(v)} \\ 0 & K_2^{(v)} \end{bmatrix}, \quad (4.18)$$

where the matrices $K_1^{(u)}$, $K_1^{(v)}$ and $K_2^{(v)}$ are of sizes $m_1 \times m_1$, $m_1 \times m_2$ and $m_2 \times m_2$ respectively. The system in (4.14) for $w = (w_1, w_2)$ now reads

$$-\operatorname{div}(P^{(u)}(u, v)Dw_1 + P^{(v)}(u, v)Dw_2) + K_1^{(u)}w_1 + K_1^{(v)}w_2 = f^{(u)}(u, v) + K_1^{(u)}u + K_1^{(v)}v, \quad (4.19)$$

and

$$-\operatorname{div}(Q^{(u)}(u, v)Dw_1 + Q^{(v)}(u, v)Dw_2) + K_2^{(v)}w_2 = f^{(v)}(u, v) + K_2^{(v)}v. \quad (4.20)$$

We will consider the following assumptions on the above subsystems.

K.0) Assume that there are $k_1, k_2 > 0$ such that

$$\langle K_1^{(u)}x_1, x_1 \rangle \geq k_1|x_1|^2, \quad \langle K_2^{(v)}x_2, x_2 \rangle \geq k_2|x_2|^2 \quad \forall x_i \in \mathbb{R}^{m_i}, i = 1, 2. \quad (4.21)$$

K.1) For all $(u, v) \in \mathbf{B} \cap \mathbf{P}$

$$\hat{f}(u, v) + K(u, v) \geq 0,$$

and the following maximum principle holds: if $(u, v) \in \mathbf{B} \cap \mathbf{P}$ and w solves

$$\begin{cases} -\operatorname{div}(A(u, v)Dw) + Kw \geq 0 & \text{in } \Omega, \\ \text{Homogenous boundary conditions} & \text{on } \partial\Omega \end{cases}$$

then $w \geq 0$.

K.2) For any $u \in Z_1$ and $\phi_2 \in \dot{\mathbf{P}}_2$ a strong maximum principle holds for the system

$$-\operatorname{div}(Q^{(v)}(u, 0)DU_2 + Q_v^{(u)}(u, 0)Du\phi_2) + K_2^{(v)}U_2 = f_v^{(v)}(u, 0)\phi_2 + K_2^{(v)}\phi_2. \quad (4.22)$$

That is, if $f_v^{(v)}(u, 0)\phi_2 + K_2^{(v)}\phi_2 \in \dot{\mathbf{P}}_2$ then $U_2 \in \dot{\mathbf{P}}_2$.

Concerning the term $Q_v^{(u)}(u, 0)Du\phi_2$ in K.2) we have used the following notation: if $B(u, v) = (b_{ij}(u, v))$, with $i = 1, \dots, m_2$ and $j = 1, \dots, m_1$, and $\phi_2 = (\phi^{(1)}, \dots, \phi^{(m_2)})$ then

$$B_v(u, v)Du\phi_2 = \left(\partial_{v^{(k)}} b_{ij}(u, v) \phi^{(k)} Du_j \right)_i = (\partial_{v^{(k)}} b_{ij}(u, v) Du_j)_{k,i} \phi_2. \quad (4.23)$$

We also assume that

K.3) For any $u \in Z_1$ the linear system

$$-\operatorname{div}(Q^{(v)}(u, 0)Dh_2 + Q_v^{(u)}(u, 0)Duh_2) = f_v^{(v)}(u, 0)h_2$$

has no positive solution h_2 in \mathbf{P}_2 .

For $u \in Z_1$ we will also consider the following eigenvalue problem for an eigenfunction h

$$-\operatorname{div}(\lambda Q^{(v)}(u, 0)Dh + Q_v^{(u)}(u, 0)Duh) + \lambda K_2^{(v)}h = f_v^{(v)}(u, 0)h + K_2^{(v)}h, \quad (4.24)$$

and denote

$$Z_1^- := \{u \in Z_1 : (4.24) \text{ has a positive solution } h \text{ to an eigenvalue } \lambda < 1\}.$$

The main theorem of this subsection is the following.

Theorem 4.12 *Assume K.0)-K.4) with k_1 in K.0) being sufficiently large. Then the map T described in (4.14) is well defined on $\mathbf{B} \cap \mathbf{P}$ and maps $\mathbf{B} \cap \mathbf{X}$ into \mathbf{P} . There are neighborhoods U, U^- respectively of Z_1, Z_1^- in \mathbf{P}_1 and a neighborhood V of 0 in \mathbf{P}_2 such that*

$$\operatorname{ind}(T, U \oplus V) = \operatorname{ind}(F_1(\cdot, 0), U^-).$$

Here, $F_1(\cdot, 0)$ maps $\mathbf{B} \cap \mathbf{P}_1$ into \mathbf{P}_1 and $w_1 = F_1(u, 0)$, $u \in \mathbf{B} \cap \mathbf{P}_1$, is the unique solution to

$$-\operatorname{div}(P^{(u)}(u, 0)Dw_1) + K_1^{(u)}w_1 = f^{(u)}(u, 0) + K_1^{(u)}u. \quad (4.25)$$

The above theorem is just a consequence of Theorem 4.1 applying to the system (4.14). We need only to verify the assumption of the theorem. For this purpose and later use in the section we will divide its proof into lemmas which also contain additional and useful facts.

We first have the following lemma which shows that the assumption (4.6), that $F_2(u, 0) = 0$ for all $u \in \mathbf{X}_1$, in the previous section is satisfied.

Lemma 4.13 *Let T be defined by (4.14). If $K.0$ holds for some sufficiently large k_1 then T is well defined by (4.14) for any given matrix $K_1^{(v)}$. The components F_1, F_2 of T satisfy*

- i) $F_2(u, 0) = 0$ for all $u \in \mathbf{X}_1$.
- ii) $w_1 = F_1(u, 0)$ solves (4.25). In addition, $\text{ind}(F_1(\cdot, 0), \mathbf{B} \cap \mathbf{P}_1) = 1$.

Proof: We write $w = T(u, v) = (F_1(u, v), F_2(u, v))$ in (4.14) by (w_1, w_2) , with $w_i \in \mathbf{X}_i$. Because

$$\langle Kw, w \rangle = \langle K_1^{(u)} w_1, w_1 \rangle + \langle K_1^{(v)} w_2, w_1 \rangle + \langle K_2^{(v)} w_2, w_2 \rangle,$$

a simple use of Young's inequality and (4.21) show that if k_1 sufficiently large then $\langle Kx, x \rangle \geq |x|^2$ for any given $K_1^{(v)}$. Hence, T is well defined by (4.14).

At $(u, 0)$, since $\hat{f}^{(v)}(u, 0) = 0$ and $Q^{(u)}(u, 0) = 0$, the subsystem (4.20) defining $w_2 = F_2(u, 0)$ now is

$$-\text{div}(Q^{(v)}(u, 0)Dw_2) + K_2^{(v)}w_2 = 0.$$

This system has $w_2 = 0$ as the only solution because of the assumption (4.21) on $K_2^{(v)}$ and the ellipticity of $Q^{(v)}(u, 0)$. This gives i).

Next, as w_2 and Dw_2 are zero, (4.19) gives that $w_1 = F_1(u, 0)$ solves

$$-\text{div}(P^{(u)}(u, 0)Dw_1) + K_1^{(u)}w_1 = f^{(u)}(u, 0) + K_1^{(u)}u.$$

Again, for a given $u \in \mathbf{X}_1$ this subsystem has a unique solution w_1 if $\langle K_1^{(u)}x, x \rangle \geq k_1|x|^2$ for some $k_1 > 0$. Moreover, the fixed point of $u = F_1(u, 0)$ solves

$$-\text{div}(P^{(u)}(u, 0)Du) = f^{(u)}(u, 0).$$

This system satisfies the same set of structural conditions for the full system (4.1) so that Theorem 3.2 can apply here to give ii). ■

From the proof of Theorem 4.1 we need study the Frechet (directional) derivative of T defined by (4.14). For this purpose and later use, we consider a more general linear system defining $w = T(u)$

$$\begin{cases} -\text{div}(A(u)Dw + B(u, Du)w) + C(u, Du)w = \hat{f}(u, Du) & x \in \Omega, \\ w = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.26)$$

for some matrix valued functions A, B, C, \hat{f} .

We then recall the following elementary result on the linearization of the above system at u .

Lemma 4.14 *Let u, ϕ be in \mathbf{X} . If $w = T(u)$ is defined by (4.26) then $W = T'(u)\phi$ solves the following system*

$$-\operatorname{div}(A(u)DW + B(u, Du)W + \mathcal{B}(u, w, \phi)) + \mathcal{C}(u, W, w, \phi) = \mathcal{F}(u, \phi),$$

where

$$\begin{aligned}\mathcal{B}(u, w, \phi) &= A_u(u)\phi Dw + B_u(u, Du)\phi w + B_\zeta(u, Du)D\phi w, \\ \mathcal{C}(u, W, w, \phi) &= C(u, Du)W + C_u(u, Du)\phi w + C_\zeta(u, Du)D\phi w, \\ \mathcal{F}(u, \phi) &= \hat{f}_u(u, Du)\phi + \hat{f}_\zeta(u, Du)D\phi.\end{aligned}$$

The proof of this lemma is standard. Because A, \hat{f} are C^1 in u , it is easy to see that T is differentiable. In fact, for any $u, \phi \in \mathbf{X}$ we can compute $T'(u)\phi = \lim_{h \rightarrow 0} \delta_{h, \phi} T(u)$, where $\delta_{h, \phi}$ is the difference quotient operator

$$\delta_{h, \phi} T(u) = h^{-1}(T(u + h\phi) - T(u)).$$

Subtracting (4.3) at u being $u + h\phi$ and u and dividing the result by h , we get

$$\begin{aligned}-\operatorname{div}(\delta_{h, \phi}[A(u)DT(u) + B(u, Du)T(u)] + \\ \delta_{h, \phi}[C(u, Du)T(u)] = \delta_{h, \phi}\hat{f}(u, Du).\end{aligned}\tag{4.27}$$

It is elementary to see that if g is a C^1 function in $u, \zeta = Du, w = T(u), Dw$ then

$$\lim_{h \rightarrow 0} \delta_{h, \phi} g(u, Du, w, Dw) = g_u \phi + g_\zeta D\phi + g_w T'(u)\phi + g_{Dw} D(T'(u)\phi).$$

Using the above in (4.27) and rearranging the terms, we obtain the lemma.

Applying Lemma 4.14 to the system (4.20), we have the following lemma concerning the map $\partial_v F_2(u, 0)$ at $u \in Z_1$.

Lemma 4.15 *Let $u \in Z_1$. An eigenvector function h of $\partial_v F_2(u, 0)h = \lambda h$ satisfies the system*

$$-\operatorname{div}(\lambda Q^{(v)}(u, 0)Dh + Q_v^{(u)}(u, 0)Duh) + \lambda K_2^{(v)}h = f_v^{(v)}(u, 0)h + K_2^{(v)}h.\tag{4.28}$$

In addition, if K.2) holds then $\partial_v F_2(u, 0)$ is strongly positive.

Proof: Let $\phi = (0, \phi_2)$ and $u \in Z_1$. We have

$$w := T(u, 0), \quad W := T'(u, 0)\phi = (\partial_v F_1(u, 0)\phi_2, \partial_v F_2(u, 0)\phi_2)$$

satisfy, by Lemma 4.14 with $B(u, Du) = 0$ and $C(u, Du) = K$

$$-\operatorname{div}(A(u, 0)DW + \partial_{u, v} A(u, 0)\phi Dw) + KW = \hat{f}_{u, v}(u, 0)\phi + K\phi.$$

At $(u, 0)$, we have that v, Dv are zero and $Dw = D(T(u, 0)) = (Du, 0)$ so that

$$A(u, 0) = \begin{bmatrix} P^{(u)}(u, 0) & P^{(v)}(u, 0) \\ 0 & Q^{(v)}(u, 0) \end{bmatrix},$$

$$\partial_{u,v}A(u,0)\phi Dw = \begin{bmatrix} P_v^{(u)}(u,0)Du\phi_2 \\ Q_v^{(u)}(u,0)Du\phi_2 \end{bmatrix}.$$

Thus, $\mathbf{U}_1 := \partial_v F_1(u,0)\phi_2$ and $\mathbf{U}_2 := \partial_v F_2(u,0)\phi_2$, the components of $T'(u,0)\phi$, satisfy

$$\begin{aligned} -\operatorname{div}(P^{(u)}(u,0)D\mathbf{U}_1 + P^{(v)}(u,0)D\mathbf{U}_2 + P_v^{(u)}(u,0)Du\phi_2) \\ + K_1^{(u)}\mathbf{U}_1 + K_1^{(v)}\mathbf{U}_2 = f_v^{(u)}(u,0)\phi_2 + K_1^{(v)}\phi_2, \end{aligned}$$

and

$$-\operatorname{div}(Q^{(v)}(u,0)D\mathbf{U}_2 + Q_v^{(u)}(u,0)Du\phi_2) + K_2^{(v)}\mathbf{U}_2 = f_v^{(v)}(u,0)\phi_2 + K_2^{(v)}\phi_2. \quad (4.29)$$

We consider the eigenvalue problem $\partial_v F_2(u,0)h = \lambda h$. Set $\phi_2 = h$ then $\mathbf{U}_2 = \lambda h$ and it is clear from (4.29) that h is the solution to (4.28).

Finally, the system (4.29) defining $\mathbf{U}_2 := \partial_v F_2(u,0)\phi_2$ is exactly (4.22) in K.2). Thus, the strong maximum principle for (4.22) yields that $\partial_v F_2(u,0)$ is strongly positive. ■

Proof of Theorem 4.12: Lemma 4.13 shows that T is well defined and maps $\mathbf{B} \cap \mathbf{P}$ into \mathbf{P} if K.1) is assumed. Lemma 4.15 and K.2) then gives the strong positivity of $\partial_v F_2(u,0)$ for any $u \in Z_1$. In addition, the equation in the condition K.3) is (4.28) of Lemma 4.15 when $\lambda = 1$ so that K.3) means that the condition Z) of the previous section holds here. Thus, our theorem is just a consequence of Theorem 4.1. ■

We now turn to semi trivial fixed points of T in \mathbf{X}_2 . These fixed points are determined by the following system, setting $w_1 = u = 0$, $w_2 = v$ in (4.19) and (4.20)

$$\begin{cases} -\operatorname{div}(P^{(v)}(0,v)Dv) = f^{(u)}(0,v), \\ -\operatorname{div}(Q^{(v)}(0,v)Dv) = f^{(v)}(0,v). \end{cases} \quad (4.30)$$

We will assume that this system has no positive solution v . In fact, if $P^{(v)}(0,v) \neq 0$ the above system is *overdetermined* so that the existence of a nonzero solution v of the second subsystem satisfying the first subsystem is very unlikely. In fact, assuming $f^{(v)}(0,0) = 0$, it could happen that the second subsystem already has $v = 0$ as the only solution.

At $(u,v) = (0,0)$, the eigenvalue problem $\partial_v F_2(0,0)h_2 = \lambda h_2$ for $h_2 \in \mathbf{X}_2$ is

$$-\operatorname{div}(Q^{(v)}(0,0)Dh_2) + K_2^{(v)}h_2 = \lambda^{-1}(f_v^{(v)}(0,0) + K_2^{(v)})h_2. \quad (4.31)$$

From ii) of Lemma 4.13, the eigenvalue problem $\partial_u F_1(0,0)h_1 = \lambda h_1$ for $h_1 \in \mathbf{X}_1$ is

$$-\operatorname{div}(P^{(u)}(0,0)Dh_1) + K_1^{(u)}h_1 = \lambda^{-1}(f_u^{(u)}(0,0) + K_1^{(u)})h_1. \quad (4.32)$$

Again, we will say that $0 \in \mathbf{X}_1$ is u -stable if the above has no positive eigenvector h_1 to any eigenvalue $\lambda > 1$. Otherwise, we say that 0 is u -unstable.

Our first application of Theorem 4.1 is to give sufficient conditions such that semi trivial and nontrivial solutions exist.

Theorem 4.16 Suppose that (4.30) has no positive solution. Assume K.0)-K.4) and that the system

$$- \operatorname{div}(P^{(u)}(0,0)Dh_1) = f_u^{(u)}(0,0)h_1 \quad \text{has no solution } h_1 \in \dot{\mathbf{P}}_1. \quad (4.33)$$

If either one of the followings holds

- i.1) 0 is u -stable and $Z_1^+ = \{0\}$;
- i.2) 0 is u -unstable and $Z_1^- = \{0\}$;

then there is a nontrivial positive solution to (4.1).

Proof: We denote $Z_p = \{u \in Z_1 : u > 0\}$. Thus Z_p is the set of semi trivial solutions and $Z_1 = \{0\} \cup Z_p$. Accordingly, we denote by Z_p^+ (resp. Z_p^-) the v -unstable (resp. v -stable) subset of Z_p . The assumption (4.33) means $\partial_u F_1(0,0)$ does not have positive eigenfunction to the eigenvalue 1 in \mathbf{P}_1 . Applying Corollary 4.9 with $X = \mathbf{X}_1$ and $F(\cdot) = F_1(\cdot, 0)$, we can find a neighborhood U_0 in \mathbf{X}_1 of 0 such that 0 is the only fixed point of $F_1(\cdot, 0)$ in U_0 and (4.13) gives

$$\operatorname{ind}(F_1(\cdot, 0), U_0) = \begin{cases} 1 & \text{if 0 is } u\text{-stable,} \\ 0 & \text{if 0 is } u\text{-unstable.} \end{cases} \quad (4.34)$$

Since Z_1 is compact and $Z_1 = \{0\} \cup Z_p$, the above argument shows that Z_p is compact. From the proof of Theorem 4.1, there are disjoint open neighborhoods U_p^- and U_p^+ in \mathbf{X}_1 of Z_p^- and Z_p^+ respectively. Of course, we can assume that U_0 , U_p^- and U_p^+ are disjoint so that for $U = U_0 \cup U_p^+ \cup U_p^-$

$$\operatorname{ind}(F_1(\cdot, 0), U) = \operatorname{ind}(F_1(\cdot, 0), U_0) + \operatorname{ind}(F_1(\cdot, 0), U_p^+) + \operatorname{ind}(F_1(\cdot, 0), U_p^-). \quad (4.35)$$

By Lemma 4.13, $\operatorname{ind}(F_1(\cdot, 0), \mathbf{B} \cap \mathbf{P}_1) = 1$. This implies $\operatorname{ind}(F_1(\cdot, 0), U) = 1$. If i.1) holds then $Z_p^+ = \emptyset$ and we can take $U_p^+ = \emptyset$ and $U^- = U_p^-$ in Theorem 4.1. From (4.34), $\operatorname{ind}(F_1(\cdot, 0), U_0) = 1$ so that (4.35) implies $\operatorname{ind}(F_1(\cdot, 0), U_p^-) = 0$. This yields $\operatorname{ind}(F_1(\cdot, 0), U^-) = 0$. Similarly, If i.2) holds then $Z_1^- = \{0\}$ and $Z_p^- = \emptyset$ and we can take $U_p^- = \emptyset$ and $U^- = U_0$ in Theorem 4.1. From (4.35), $\operatorname{ind}(F_1(\cdot, 0), U^-) = \operatorname{ind}(F_1(\cdot, 0), U_0) = 0$.

Hence, $\operatorname{ind}(F_1(\cdot, 0), U^-) = 0$ in both cases. By Theorem 4.12, we find a neighborhood V of 0 in \mathbf{P}_2 such that $\operatorname{ind}(T, U \oplus V) = \operatorname{ind}(F_1(\cdot, 0), U^-) = 0$. Since $\operatorname{ind}(T, \mathbf{B} \cap \mathbf{X}) = 1$, we see that T has a fixed point in $\mathbf{B} \setminus \overline{U \oplus V}$. This fixed point is nontrivial because we are assuming that T has no semi trivial fixed point in \mathbf{P}_2 . ■

4.3 Notes on a more special case and a different way to define T :

In many applications, it is reasonable to assume that the cross diffusion effects by other components should be proportional to the density of a given component. This is to say that if $A(u) = (a_{ij}(u))$ then there are smooth functions b_{ij} such that

$$a_{ij}(u) = u_i b_{ij}(u) \quad \text{if } j \neq i. \quad (4.36)$$

In this case, instead of using (4.14), we can define the i -th component w_i of $T(u)$ by

$$L_i(u)w_i + \sum_j k_{ij}w_j = f_i(u) + \sum_j k_{ij}u_j, \quad (4.37)$$

where

$$L_i(u)w = -\operatorname{div}(a_{ii}(u)Dw + w(\sum_{j \neq i} b_{ij}(u)Du_j)). \quad (4.38)$$

As $u \in \mathbf{B} \cap \mathbf{P}$, (4.37) is a *weakly coupled* system with Hölder continuous coefficients. We will see that the condition K.1) on the positivity of solutions in the previous section is verified. To this end, we recall the maximum principles for cooperative linear systems in [19, 20] and give here an alternative and simple proof to [19, Theorem 1.1]. In fact, we consider a more general setting that covers both Dirichlet and Neumann boundary conditions.

Let us define

$$L_i w = -\operatorname{div}(\alpha_i(x)Dw + \beta_i(x)w), \quad (4.39)$$

where $\alpha_i \in L^\infty(\Omega)$, $\beta_i \in L^\infty(\Omega, \mathbb{R}^n)$. Denote $\mathcal{F} = (\mathcal{F}_1, \dots, \mathcal{F}_m)$. We then have the following weak minimum principle.

Lemma 4.17 *Let w be a weak solution to the system*

$$\begin{cases} L_i w_i + K w = \mathcal{F}_i, & i = 1, \dots, m, \text{ in } \Omega, \\ \text{Homogeneous Dirichlet or Neumann boundary conditions on } \partial\Omega. \end{cases}$$

Assume that $\alpha_i(x) \geq \lambda_i$ for some $\lambda_i > 0$ and $\mathcal{F}_i \geq 0$ for all i . If $k_{ij} \leq 0$ for $i \neq j$ and k_{ii} are sufficiently large, in terms of $\sup_\Omega \beta_i(u(x))$, then $w \geq 0$.

Proof: Let ϕ^+, ϕ^- denote the positive and negative parts of a scalar function ϕ , i.e. $\phi = \phi^+ - \phi^-$. We note that $\langle Dw_i, Dw_i^- \rangle = -|Dw_i^-|^2$ and $w_i Dw_i^- = -w_i^- Dw_i^-$. Integrating by parts, we have

$$\int_\Omega L_i w_i w_i^- dx = \int_\Omega (-\alpha_i |Dw_i^-|^2 - \langle \beta_i, Dw_i^- \rangle w_i^-) dx$$

Hence, multiplying the i -th equation of the system by $-w_i^-$, we obtain

$$\sum_i \int_\Omega (\alpha_i |Dw_i^-|^2 + \langle \beta_i, Dw_i^- \rangle w_i^-) dx - \sum_{i,j} \int_\Omega k_{ij} w_i w_j^- dx = - \sum_i \int_\Omega \langle \mathcal{F}_i, w_i^- \rangle dx.$$

Since $\mathcal{F}_i, w_i^- \geq 0$, we get

$$\sum_i \int_\Omega (\alpha_i |Dw_i^-|^2 + \langle \beta_i, Dw_i^- \rangle w_i^-) dx - \sum_{i,j} \int_\Omega k_{ij} w_i w_j^- dx \leq 0.$$

Since $w_i w_i^- = -w_i^- w_i^-$ and $w_i w_j^- = w_i^+ w_j^- - w_i^- w_j^- \geq w_i^- w_j^-$, the above yields, using the assumption that $k_{ij} \leq 0$ for $i \neq j$

$$\sum_i \int_\Omega (\alpha_i |Dw_i^-|^2 + \langle \beta_i, Dw_i^- \rangle w_i^- + k_{ii} |w_i^-|^2) dx - \sum_{i \neq j} \int_\Omega k_{ij} w_i^- w_j^- dx \leq 0. \quad (4.40)$$

By Young's inequality, for any $\varepsilon > 0$ we can find a constant $C(\varepsilon, \beta_i)$, depending on $\sup_{\Omega} \beta_i(u(x))$, such that

$$\left| \int_{\Omega} \langle \beta_i, Dw_i^- \rangle w_i^- dx \right| \leq \varepsilon \int_{\Omega} |Dw_i^-|^2 dx + C(\varepsilon, \beta_i) \int_{\Omega} |w_i^-|^2 dx.$$

Thus, (4.40) implies

$$\sum_i \int_{\Omega} (\alpha_i - \varepsilon) |Dw_i^-|^2 dx + \sum_i (k_{ii} - C(\varepsilon, \beta_i)) \int_{\Omega} |w_i^-|^2 dx - \sum_{i \neq j} \int_{\Omega} k_{ij} w_i^- w_j^- dx \leq 0.$$

Combining the ellipticity assumption and Poincaré's inequality, we have

$$c_0(\lambda_i - \varepsilon) \int_{\Omega} |w_i^-|^2 dx \leq \int_{\Omega} (\alpha_i - \varepsilon) |Dw_i^-|^2 dx$$

for some $c_0 > 0$. Therefore,

$$c_0(\lambda_i - \varepsilon) \sum_i \int_{\Omega} |w_i^-|^2 dx + \sum_i (k_{ii} - C(\varepsilon, \beta_i)) \int_{\Omega} |w_i^-|^2 dx - \sum_{i \neq j} \int_{\Omega} k_{ij} w_i^- w_j^- dx \leq 0.$$

This implies

$$\int_{\Omega} \sum_{i,j} \gamma_{ij} w_i^- w_j^- dx \leq 0, \quad (4.41)$$

where

$$\gamma_{ij} = \begin{cases} c_0(\lambda_i - \varepsilon) + k_{ii} - C(\varepsilon, \beta_i) & i = j, \\ -k_{ij} & i \neq j. \end{cases}$$

It is clear that if k_{ii} is sufficiently large then the matrix $\gamma = (\gamma_{ij})$ is positive definite, i.e. $\langle \gamma x, x \rangle \geq c|x|^2$ for some positive c . Thus, (4.41) forces $w_i^- = 0$ a.e. and $w \geq 0$. ■

Thanks to this lemma, we now see how to construct a matrix K such that T maps $\mathbf{B} \cap \mathbf{P}$ into \mathbf{P} . To this end, we note that $f^{(u)}(0, 0) = 0$ and $f^{(v)}(u, 0) = 0$ so that we can write

$$f^{(u)}(u, v) + K_1^{(u)} u + K_1^{(v)} v = \int_0^1 (f_u^{(u)}(tu, tv) + K_1^{(u)}) dt u + \int_0^1 (f_v^{(u)}(tu, tv) + K_1^{(v)}) dt v,$$

$$f^{(v)}(u, v) + K_2^{(v)} v = \int_0^1 (f_v^{(v)}(u, tv) + K_2^{(v)}) dt v.$$

Since $\|\hat{f}_u(u)\|_{\mathbf{X}}$ is bounded for $u \in \mathbf{B} = B(0, M)$ (the bound M is independent of K), it is not difficult to see that if the reaction is 'cooperative', i.e., $\partial_{u_i} \hat{f}_j \geq 0$ for $i \neq j$, then we can always find K with $k_{ij} = 0$ for $i \neq j$ and $k_{ii} > 0$ sufficiently large such that the matrix integrands in the above equations are all positive. Therefore, $\mathcal{F} := \hat{f}(u, v) + K(u, v) \geq 0$ for $(u, v) \in \mathbf{B} \cap \mathbf{P}$ and the lemma can apply here.

Finally, for future use in the next section we now explicitly describe the map $T'(u)$ in this case. For $\Phi = (\phi_1, \dots, \phi_m)$, by Lemma 4.14, the components $w_i = T_i(u)$ of $T(u)$ and $W_i = T'_i(u)\Phi$ of $T'(u)\Phi$ solves

$$-\operatorname{div}(\mathcal{A}_i(u, W_i) + \mathcal{B}_i(u, w_i, \Phi)) + \sum_j k_{ij} W_j = \sum_j (\partial_{u_j} f_i(u) + k_{ij}) \phi_j, \quad (4.42)$$

where

$$\mathcal{A}_i(u, W_i) := a_{ii}(u) DW_i + \left(\sum_{j \neq i} b_{ij}(u) Du_j \right) W_i, \quad (4.43)$$

$$\mathcal{B}_i(u, w_i, \Phi) := \sum_j \partial_{u_j} a_{ii}(u) \phi_j Dw_i + w_i \sum_{j \neq i} [\partial_{u_k} b_{ij}(u) \phi_k Du_j + b_{ij}(u) D\phi_j]. \quad (4.44)$$

Consider a semi trivial solution $u \in Z_1$, i.e. for some integer $m_1 \geq 0$

$$T(u, 0) = (u, 0), \quad (u, 0) = (u_1, \dots, u_{m_1}, 0, \dots, 0).$$

For $i > m_1$ we have that $w_i = T_i(u, 0)$ and $Dw_i = D(T_i(u, 0))$ are zero so that W_i solves

$$-\operatorname{div}(a_{ii}(u, 0) DW_i + \sum_{j \leq m_1} b_{ij}(u, 0) Du_j W_i) + \sum_j k_{ij} W_j = \sum_j (\partial_{u_j} f_i(u, 0) + k_{ij}) \phi_j. \quad (4.45)$$

5 Nonconstant and Nontrivial Solutions

We devote this section to the study of (4.1) with Neumann boundary conditions. Theorem 4.16 gives the existence of positive nontrivial solution but this solution may be a constant solution. This is the case when there is a constant vector $u^* = (u_1^*, \dots, u_m^*)$ such that $f(u^*) = 0$. Obviously $u = u_*$ is a nontrivial solution to (4.1) and Theorem 4.16 then yields no useful information. In applications, we are interested in finding a nonconstant solution besides this obvious solution. We will assume throughout this section that the semi trivial solutions are all constant and show that cross diffusion will play an important role for the existence of nonconstant and nontrivial solutions.

Inspired by the SKT systems, we assume that the diffusion is given by (4.36) as in Section 4.3 and the reaction term in the i -th equation is also proportional to the density u_i . This means,

$$f_i(u) = u_i g_i(u) \quad (5.1)$$

for some C^1 functions g_i 's. A constant solution u^* exists if it is a solution to the equations $g_i(u^*) = 0$ for all i .

Throughout this section, we denote by ψ_i 's the eigenfunctions of $-\Delta$, satisfying Neumann boundary condition, to the eigenvalue $\hat{\lambda}_i$ such that $\{\psi_i\}$ is a basis for $W^{1,2}(\Omega)$. That is,

$$\begin{cases} -\Delta \psi = \hat{\lambda}_i \psi \text{ in } \Omega, \\ \psi \text{ satisfies homogeneous Neumann boundary condition.} \end{cases}$$

5.1 Semi trivial constant solutions

We consider a semi trivial solution $(u, 0)$ with $u = (u_1, \dots, u_{m_1})$ for some integer $m_1 = 0, \dots, m$. Following the analysis of Section 4.1, we need to consider the eigenvalue problem

$\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ with $\Phi_2 = (\phi_{m_1}, \dots, \phi_m)$ and $\Phi = (0, \Phi_2)$. Then the equation (4.45), with $W_i = \mu\phi_i$ for $i = m_1 + 1, \dots, m$, gives

$$-\operatorname{div}(a_{ii}(u, 0)D\phi_i + \sum_{j \leq m_1} b_{ij}(u, 0)Du_j\phi_i) + \sum_{j > m_1} k_{ij}\phi_j = \mu^{-1} \sum_{j > m_1} (\partial_{u_j} f_i(u, 0) + k_{ij})\phi_j.$$

If u is a constant vector then $Du_j = 0$ and the above reduces to

$$-\operatorname{div}(a_{ii}(u, 0)D\phi_i) + \sum_{j > m_1} k_{ij}\phi_j = \mu^{-1} \sum_{j > m_1} (\partial_{u_j} f_i(u, 0) + k_{ij})\phi_j, \quad (5.2)$$

which is an elliptic system with constant coefficients. We then need the following lemma.

Lemma 5.1 *Let A, B be constant matrices. Then the solution space of the problem*

$$\begin{cases} -\operatorname{div}(AD\Phi) = B\Phi, \\ \text{Neumann boundary conditions.} \end{cases}$$

has a basis $\{\mathbf{c}_{i,j}\psi_i\}$ where $\mathbf{c}_{i,j}$'s are the basis vectors of $\operatorname{Ker}(\hat{\lambda}_i A - B)$.

The proof of this lemma is elementary. If Φ solves its equation of the lemma then we can write $\Phi = \sum k_i \psi_i$, in $W^{1,2}(\Omega)$, with $k_i \in \mathbb{R}^m$. We then have $\sum_i \hat{\lambda}_i A k_i \psi_i = \sum_i B k_i \psi_i$. Since $\{\psi_i\}$ is a basis of $W^{1,2}(\Omega)$, this equation implies $\hat{\lambda}_i A k_i = B k_i$ for all i . Thus, k_i is a linear combination of $\mathbf{c}_{i,j}$'s. It is easy to see that $\{\mathbf{c}_{i,j}\psi_j\}$ is linearly independent if $\{\mathbf{c}_{i,j}\}, \{\psi_j\}$ are. The lemma then follows.

Lemma 5.2 *Let m_1 be a nonnegative integer less than m and $u = (u_1, \dots, u_{m_1})$ be a constant function such that $T(u, 0) = (u, 0)$. Then*

$$\partial_v f^{(v)}(u, 0)\mathbf{c} = \lambda K_2^{(v)}\mathbf{c}$$

has a positive eigenvector \mathbf{c} to a positive (respectively, negative) eigenvalue λ if and only if the eigenvalue problem $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ has a positive solution for some $\mu > 1$ (respectively, $\mu < 1$).

Proof: By (5.2), the eigenvalue problem $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ (or $W_i = \mu\phi_i$) is determined by the following system

$$-\mu \operatorname{div}(A^{(m_1)}(u, 0)D\Phi_2) + \mu K_2^{(v)}\Phi_2 = [\partial_v f^{(v)}(u, 0) + K_2^{(v)}]\Phi_2, \quad (5.3)$$

where $A^{(m_1)}(u, 0) = \operatorname{diag}[a_{ii}(u, 0)]_{i > m_1}$. The coefficients of the above system are constant and Lemma 5.1 yields that the solutions to the above is $\sum \mathbf{c}_i \psi_i$ with \mathbf{c}_i solving

$$\mu[\hat{\lambda}_i A^{(m_1)}(u, 0) + K_2^{(v)}]\mathbf{c} = [\partial_v f^{(v)}(u, 0) + K_2^{(v)}]\mathbf{c}.$$

Note that the only positive eigenfunction of $-\Delta$ is $\psi_0 = 1$ to the eigenvalue $\hat{\lambda}_0 = 0$. Therefore, from the above system with $i = 0$ we see that if the system

$$\mu K_2^{(v)}\mathbf{c} = [\partial_v f^{(v)}(u, 0) + K_2^{(v)}]\mathbf{c} \Leftrightarrow \partial_v f^{(v)}(u, 0)\mathbf{c} = (\mu - 1)K_2^{(v)}\mathbf{c}$$

has a positive solution \mathbf{c} then the constant function \mathbf{c} is a positive eigenfunction for $\partial_v F_2(u, 0)$. Conversely, if $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ has a positive solution Φ_2 then we integrate (5.3) over Ω , using the Neumann boundary conditions, to see that $\mathbf{c} = \int_{\Omega} \Phi_2 dx$ is a positive solution to the above system. The lemma then follows. ■

Remark 5.3 By the Krein-Ruthman theorem, if $\partial_v F_2(u, 0)$ is strongly positive then $\mu = r_v(u)$ is the only eigenvalue with positive eigenvector. The eigenvalue problem $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ has a positive solution for $\mu = 1$ if and only if the matrix $\partial_v f^{(v)}(u, 0)$ has a positive eigenvector to the zero eigenvalue.

We now discuss the special case $f_i(u) = u_i g_i(u)$.

Lemma 5.4 *Assume that $f_i(u) = u_i g_i(u)$. Let m_1 be a nonnegative integer less than m and $u = (u_1, \dots, u_{m_1})$ be a constant vector such that $T(u, 0) = (u, 0)$. Then the eigenvalue problem $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$*

- i.1)** *has no nonzero solution for $\mu = 1$ if and only if $g_i(u, 0) \neq 0$ for any $i > m_1$;*
- i.2)** *has a positive solution for some $\mu > 1$ if and only if $g_i(u, 0) > 0$ for some $i > m_1$;*
- i.3)** *has no positive solution for $\mu > 1$ if and only $g_i(u, 0) < 0$ for any $i > m_1$.*

Proof: We now let $K = kI$. By Lemma 5.2 the existence of positive eigenvectors of $\partial_v F_2(u, 0)\Phi_2 = \mu\Phi_2$ is equivalent to that of

$$\partial_v f^{(v)}(u, 0)\mathbf{c} = k\lambda\mathbf{c} \quad \text{with } \mathbf{c} = (c_{m_1+1}, \dots, c_m) > 0 \text{ and } \lambda = \mu - 1. \quad (5.4)$$

Since $f_i(u) = u_i g_i(u)$ and $u_i = 0$ for $i > m_1$, we have $\partial_{u_k} f_i(u, 0) = \delta_{ik} g_i(u, 0)$, where δ_{ik} is the Kronecker symbol, for $i, k > m_1$. Thus, $\partial_v f^{(v)}(u, 0)$ is a diagonal matrix and (5.4) is simply

$$g_i(u, 0)c_i = k\lambda c_i \quad \forall i > m_1.$$

Clearly i.1) holds because then the above system has nonzero eigenvector to $\lambda = 0$. For i.2) we can take $\lambda = g_i(u, 0)/k > 0$ and $c_i = 1$, other components of \mathbf{c} can be zero. i.3) is obvious. The proof is complete. ■

We then have the following theorem for systems of two equations.

Theorem 5.5 *Assume that $f_i(u) = u_i g_i(u)$ for $i \in \{1, 2\}$. Suppose that the trivial and semi trivial solutions are only the constant ones $(0, 0)$, $u_{1,*}$ and $u_{2,*}$. This means, $g_i(u_{i,*}) = 0$. Then there is a nontrivial solution $(u_1, u_2) > 0$ in the following situations:*

- a)** $g_i(0) > 0$, $i = 1, 2$, and $g_1(u_{2,*})$ and $g_2(u_{1,*})$ are positive.
- b)** $g_i(0) > 0$, $i = 1, 2$, and $g_1(u_{2,*})$ and $g_2(u_{1,*})$ are negative.
- c)** $g_1(0) > 0$, $g_2(0) < 0$, and $g_2(u_{1,*}) > 0$.

Proof: We just need to compute the local indices of T at the trivial and semi trivial solutions and show that the sum of these indices is not 1.

First of all, by i.1) of Lemma 5.4, it is clear that the condition Z) at these solutions are satisfied in the above situations.

The conditions in case a) and i.2) of Lemma 5.4 imply that 0 and the semi trivial solutions are unstable in their complement directions. Theorem 4.12, with $Z_1^- = \emptyset$, gives that the local indices at these solutions are all zero. Similarly, in case b), the local index at 0 is 0 and the local indices at the semi trivial solutions, which are stable in their complement directions, are 1. In these cases, the sum of the indices is either 0 or 2.

In case c), because $g_2(0) < 0$ we see that 0 is u_2 -stable so that $T_2 := T|_{\mathbf{X}_2}$, the restriction of the map T to \mathbf{X}_2 , has its local index at 0 equal 1 and therefore its local index at $u_{2,*}$ is zero (see also the proof of Theorem 4.16). The assumption $g_1(0) > 0$ also yields a neighborhood V_1 in \mathbf{X}_1 of 0 such that $\text{ind}(T, V_1 \oplus \mathbf{X}_2) = 0$ (the stability of $u_{2,*}$ in the u_1 direction does not matter). On the other hand, because $g_1(0) > 0$ we see that 0 is u_1 -unstable so that $T_1 := T|_{\mathbf{X}_1}$, the restriction of the map T to \mathbf{X}_1 , has its local index at 0 equal 0 and therefore its local index at $u_{1,*}$ is 1. But $u_{1,*}$ is u_2 -unstable, because $g_2(u_{1,*}) > 0$, so that there is a neighborhood V_2 in \mathbf{X}_2 of 0 such that $\text{ind}(T, \mathbf{X}_1 \oplus V_2) = 0$.

In three cases, we have shown that the sum of the local indices at the trivial and semi trivial solutions is not 1. Hence, there is a positive nontrivial fixed point (u_1, u_2) . ■

Remark 5.6 If the system $g_i(u) = 0$, $i = 1, 2$, has no positive constant solution then the above theorem gives conditions for the existence of nonconstant and nontrivial solutions. This means pattern formations occur.

5.2 Nontrivial constant solutions

Suppose now that $u_* = (u_1, \dots, u_m)$ is a nontrivial constant fixed point of T with $u_i \neq 0$ for all i . We will use the Leray Schauder theorem to compute the local index of T at u_* . Since u_* is in the interior of \mathbf{P} , we do not need that T is positive as in the previous discussion so that we can take $K = 0$. The main result of this section, Theorem 5.8, yields a formula to compute the indices at nontrivial constant fixed points. In applications, the sum of these indices and those at semi trivial fixed points will shows the possibility of nontrivial and nonconstant fixed points to exist.

In the sequel, we will denote

$$d_A(u_*) = \text{diag}[a_{11}(u_*), \dots, a_{mm}(u_*)]. \quad (5.5)$$

From the ellipticity assumption on A , we easily see that $a_{ii}(u_*) > 0$ for all i and thus $d_A(u_*)$ is invertible.

The following lemma describes the eigenspaces of $T'(u_*)$.

Lemma 5.7 *The solution space of $T'(u_*)\Phi = \mu\Phi$ is spanned by $\mathbf{c}_i\psi_i$ with \mathbf{c}_i solving*

$$\hat{\lambda}_i[A(u_*) + (\mu - 1)d_A(u_*)]\mathbf{c}_i = \partial_u F(u_*)\mathbf{c}_i. \quad (5.6)$$

Proof: We have $D(T(u_*)) = Du_* = 0$ so that (4.42)-(4.44), with $w = u_*$, show that the i -th component W_i of $T'(u_*)\Phi$ solves

$$-\operatorname{div}(a_{ii}(u_*)DW_i + w_i \sum_{j \neq i} b_{ij}(u_*)D\phi_j) = \partial_{u_k} f_i(u_*)\phi_k.$$

Since $w_i b_{ij}(u_*) = u_i b_{ij}(u_*) = a_{ij}(u_*)$, the eigenvalue problem $W = T'(u_*)\Phi = \mu\Phi$ is

$$-\operatorname{div}(a_{ii}(u_*)D(\mu\phi_i) + \sum_{j \neq i} a_{ij}(u_*)D\phi_j) = \partial_{u_k} f_i(u_*)\phi_k.$$

In matrix form, the above can be written as

$$-\operatorname{div}([A(u_*) + (\mu - 1)d_A(u_*)]D\Phi) = \partial_u F(u_*)\Phi. \quad (5.7)$$

Since u_* is a constant vector, by Lemma 5.1 we can write $\Phi = \sum \mathbf{c}_i \psi_i$ with \mathbf{c}_i solving

$$\hat{\lambda}_i[A(u_*) + (\mu - 1)d_A(u_*)]\mathbf{c}_i = \partial_u F(u_*)\mathbf{c}_i \quad \forall i.$$

This is (5.6) and the lemma is proved. ■

We now have the following explicit formula for $\operatorname{ind}(T, u_*)$.

Theorem 5.8 *Assume that*

$$\operatorname{Ker}(\hat{\lambda}_i A(u_*) - \partial_u F(u_*)) = \{0\} \quad \forall i. \quad (5.8)$$

For $i > 0$ let $\mathcal{A}_i = A(u_*) - \hat{\lambda}_i^{-1} \partial_u F(u_*)$ and

$$N_i = \sum_{\lambda < 0} \dim(\operatorname{Ker}(d_A(u_*)^{-1} \mathcal{A}_i - \lambda I)).$$

We also denote by M_i the multiplicity of $\hat{\lambda}_i$.

Then there exists an integer L_0 such that

$$\operatorname{ind}(T, u_*) = (-1)^\gamma, \quad \gamma = \sum_{i \leq L_0} N_i M_i. \quad (5.9)$$

Proof: We will apply Leray-Schauder's theorem to compute $\operatorname{ind}(T, u_*)$. First of all, Lemma 5.7 and (5.8) show that $\Phi = 0$ is the only solution to $T'(u_*)\Phi = \Phi$ so that $\mu = 1$ is not an eigenvalue of $T'(u_*)$.

By Leray-Schauder's theorem, we have that $\operatorname{ind}(T, u_*) = (-1)^\gamma$, where γ is the sum of multiplicities of eigenvalues μ of $T'(u_*)$ which are greater than 1. Lemma 5.7 then clearly shows that γ is the sum of the dimensions of solution spaces of (5.6) and

$$\gamma := \sum_i \gamma_i^* M_i, \quad (5.10)$$

and $\gamma_i^* = \sum_{\mu > 1} n_{i,\mu}$, where $n_{i,\mu}$ is the dimension of the solution space of

$$\hat{\lambda}_i[A(u_*) + (\mu - 1)d_A(u_*)]\mathbf{c} - \partial_u F(u_*)\mathbf{c} = 0.$$

For $i = 0$ we have $\hat{\lambda}_0 = 0$ so that $n_{0,\mu} = \dim(\text{Ker}(\partial_u F(u_*)))$, which is zero because of (5.8). For $i > 0$ and $\mu > 1$ let $\mathcal{A}_i = A(u_*) - \hat{\lambda}_i^{-1} \partial_u F(u_*)$, as being defined in this theorem, and $\lambda = 1 - \mu < 0$. The above equation can be rewritten as $\mathcal{A}_i \mathbf{c} = \lambda d_A(u_*) \mathbf{c}$ so that \mathbf{c} is an eigenvector to a negative eigenvalue λ of the matrix $d_A(u_*)^{-1} \mathcal{A}_i$. It is clear that the number γ_i^* in (5.10) is the sum of the dimensions of eigenspaces of $d_A(u_*)^{-1} \mathcal{A}_i$ to negative eigenvalues. That is $\gamma_i^* = N_i$.

From the ellipticity assumption on A , it is not difficult to see that $d_A(u_*)^{-1} A(u_*)$ is positive definite. Therefore, $d_A(u_*)^{-1} \mathcal{A}_i$ is positive if $\hat{\lambda}_i$ is large. Thus, as $\lim_{i \rightarrow \infty} \hat{\lambda}_i = \infty$, there is an integer L_0 such that $d_A(u_*)^{-1} \mathcal{A}_i$ has no negative eigenvalues if $i > L_0$. Hence, $\gamma_i^* = N_i = 0$ if $i > L_0$. The theorem then follows. ■

Remark 5.9 Since $\hat{\lambda}_0 = 0$, (5.8) implies that $\text{Ker}(\partial_u F(u_*)) = \{0\}$ so that u_* is an isolated constant solution to $F(u_*) = 0$. Also, as $A(u_*)$ is positive definite and $\lim_{i \rightarrow \infty} \hat{\lambda}_i = \infty$, we see that (5.8) is true when i is large.

Combining Theorem 5.8 and Theorem 5.5, we obtain

Corollary 5.10 *Assume as in Theorem 5.5. Suppose further that there is only one nontrivial constant solution u_* . Let γ be as in (5.9). There is a nontrivial nonconstant solution $(u_1, u_2) > 0$ in the following situations:*

- 1) γ is odd and a) or c) of the theorem hold.
- 2) γ is even and b) of the theorem holds.

Proof: We have seen from the proof of Theorem 5.5 that the sum of the local indices at the trivial and semi trivial solutions is 0 in the cases a) and c). If γ is odd then $\text{ind}(T, u_*) = -1$. Similarly, if b) holds then the sum of the indices at the trivial and semi trivial solutions is 2. If γ is even then $\text{ind}(T, u_*) = 1$. Thus, the sum of the local indices at constant solutions is not 1 in both cases. Since $\text{ind}(T, \mathbf{B} \cap \mathbf{X}) = 1$, a nonconstant and nontrivial solution must exist. ■

5.3 Some nonexistent results

We conclude this paper by some nonexistence results showing that if the parameter λ_0 is sufficiently large then there is no nonconstant solutions. We consider the following system

$$\begin{cases} -\text{div}(A(u, Du)) = f(u) + B(u, Du) & \text{in } \Omega, \\ \text{homogenous Neumann boundary conditions} & \text{on } \partial\Omega, \end{cases} \quad (5.11)$$

We first have the following nonexistent result under a strong assumption on the uniform boundedness of solutions. This assumption will be relaxed later in Corollary 5.13.

Theorem 5.11 *Assume that A satisfies A) and $\hat{f}(u, Du) := f(u) + B(u, Du)$ for some $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ and $B : \mathbb{R}^m \times \mathbb{R}^{mn} \rightarrow \mathbb{R}^m$ such that $|B(u, p)| \leq b(u)|p|$ for some continuous*

nonnegative function b on \mathbb{R}^m . Suppose also that there is a constant C independent of λ_0 such that for any solutions of (5.11)

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

If the constant λ_0 in A) is sufficiently large then there is no nonconstant solution to (5.11).

Proof: For any function g on Ω let us denote the average of g over Ω by g_Ω . That is,

$$g_\Omega = \frac{1}{|\Omega|} \int_\Omega g \, dx.$$

Integrating (5.11) and using Neumann boundary conditions, we have $f(u)_\Omega + B(u, Du)_\Omega = 0$. Thanks to this, we test the system with $u - u_\Omega$ to get

$$\int_\Omega \langle A(u, Du), Du \rangle \, dx = \int_\Omega [\langle f(u) - f(u)_\Omega, u - u_\Omega \rangle - \langle B(u, Du)_\Omega, u - u_\Omega \rangle] \, dx. \quad (5.12)$$

We estimate the terms on the right hand side. First of all, by Hölder's inequality

$$\int_\Omega \langle f(u) - f(u)_\Omega, u - u_\Omega \rangle \, dx \leq \left(\int_\Omega |f(u) - f(u)_\Omega|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |u - u_\Omega|^2 \, dx \right)^{\frac{1}{2}}.$$

Applying Poincaré's inequality to the functions $f(u), u$ on the right hand side of the above inequality, we can bound it by

$$\text{diam}(\Omega)^2 \left(\int_\Omega |f_u(u)|^2 |Du|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega |Du|^2 \, dx \right)^{\frac{1}{2}} \leq F_* \text{diam}(\Omega)^2 \|Du\|_{L^2(\Omega)}^2,$$

where $F_* := \sup_\Omega |f_u(u(x))|$. This number is finite because we are assuming that $\|u\|_{L^\infty(\Omega)}$ is bounded uniformly.

Similarly, we define $B_* := \sup_\Omega b(u(x))$. Using the facts that $|B(u, Du)| \leq b(u)|Du| \leq B_*|Du|$, we have $|B(u, Du)_\Omega| \leq B_*|\Omega|^{-1}\|Du\|_{L^1(\Omega)}$. Furthermore, by Hölder's and Poincaré's inequalities

$$\|Du\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}}\|Du\|_{L^2(\Omega)}, \quad \|u - u_\Omega\|_{L^1(\Omega)} \leq C|\Omega|^{\frac{1}{2}}\text{diam}(\Omega)\|Du\|_{L^2(\Omega)}.$$

We then obtain

$$\left| \int_\Omega \langle B(u, Du)_\Omega, u - u_\Omega \rangle \, dx \right| \leq B_* \text{diam}(\Omega) \|Du\|_{L^2(\Omega)}^2.$$

Using the above estimates and the ellipticity condition A) in (5.12), we get

$$\lambda_0 \int_\Omega |Du|^2 \, dx \leq F_* \text{diam}(\Omega)^2 \int_\Omega |Du|^2 \, dx + B_* \text{diam}(\Omega) \int_\Omega |Du|^2 \, dx.$$

If λ_0 is sufficiently large then the above inequality clearly shows that $\|Du\|_{L^2(\Omega)} = 0$ and thus u must be a constant vector. ■

Remark 5.12 If we assume Dirichlet boundary conditions and $f(0) \equiv 0$ then 0 is the only solution if λ_0 is sufficiently large. To see this we test the system with u and repeat the argument in the proof.

The assumption on the boundedness of the L^∞ norms of the solutions in Theorem 5.11 can be weakened if $\lambda(u)$ has a polynomial growth. We have the following result.

Corollary 5.13 *The conclusion of Theorem 5.11 continues to hold for the system if one has a uniform estimate for $\|u\|_{W^{1,2}(\Omega)}$ and $\lambda(u) \sim \lambda_0 + (1 + |u|)^k$ for some $k > 0$.*

Proof: We just need to show that the two assumptions in fact provide the uniform bound of L^∞ norms needed in the previous proof. From the growth assumption on λ , we see that the number \mathbf{A} in (2.2) of Section 2 is now

$$\mathbf{A} = \sup_{W \in \mathbb{R}^m} \frac{|\lambda_W(W)|}{\lambda(W)} \sim \sup_{W \in \mathbb{R}^m} \frac{(1 + |W|)^{k-1}}{\lambda_0 + (1 + |W|)^k}.$$

By considering the cases $(1 + |W|)^k$ is greater or less than λ_0 , we can easily see that \mathbf{A} can be arbitrarily small if λ_0 is sufficiently large. On the other hand, our assumptions yield that $\|u\|_{L^1(\Omega)}$ is uniformly bounded. Thus, we can fix a $R_0 > 0$ and use the fact that $\|u\|_{BMO(B_{R_0})} \leq C(R_0)\|u\|_{L^1(\Omega)}$ to see that the condition D) of Proposition 2.1 holds if λ_0 is sufficiently large. We then have that the Hölder norms, and then L^∞ norms, of the solutions to (5.11) are uniformly bounded, independently of λ_0 . This is the key assumption of the proof of Theorem 5.11 so that the proof can continue as before. ■

Remark 5.14 From the examples in Section 3.2, e.g. Lemma 3.8 and Corollary 3.9, we see that the assumption on uniform bound for $W^{1,2}(\Omega)$ norms can be further weakened by the same assumption on $L^1(\Omega)$ norms.

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